Using Math Games and Word Problems to Increase the Math Maturity of K-8 Students

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A few highlights of his professional career include founding the International Society for Technology in Education (ISTE), serving as its executive officer for 19 years, establishing ISTE's flagship publication, *Learning and Leading with Technology*, serving as the Editor in Chief for more than 25 years. He was a major professor or co-major professor for more than 75 doctoral students. Six of these were in mathematics and the rest in education. Dr. Moursund has authored or coauthored more than 50 academic books and hundreds of articles. He has presented several hundred keynote speeches, talks, and workshops around the world. More recently, he founded Information Age Education (IAE), a non-profit organization dedicated to improving teaching and learning by people of all ages and throughout the world.

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Albrecht has authored or coauthored over 30 books and more than 150 articles, including many books about BASIC and educational games. Along with Dennis Allison, he established *Dr. Dobb's Journal*, a professional journal of software tools for advanced computer programmers. He was involved in establishing organizations, publications, and events such as Portola Institute, ComputerTown USA, *Calculators/Computers Magazine*, and the Learning Fair at Peninsula School in Menlo Park, California (now called the Peninsula School Spring Fair).

Albrecht's current adventures include writing and posting instructional materials on the Internet, writing Kindle books, tutoring high school and college students in math and physics, and running HurkleQuest play-by-email games for Oregon teachers and their students.

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Table of Contents

Preface and Introduction 3
Chapter 1: WordsWorth Games 9
Chapter 2: Introduction to Math Maturity 19
Chapter 3: Introduction to Math Intelligence 36
Chapter 4: Math Cognitive Development 49
Chapter 5: The Language of Mathematics 60
Chapter 6: Some Learning Theory 69
Chapter 7: Math Word Problems 78
Chapter 8: Math Games and Puzzles 92
Chapter 9: Dice, Coins, and Chance 105
Chapter 10: Place Value Games 118
Chapter 11: Word Problems Using Dominoes 138
Chapter 12: Factor Monster 150
Chapter 13: The Game of Pig 159
Chapter 14: More Games and Puzzles 173
Chapter 15: Final Remarks 186
Appendix 1: Make It & Take It, and Blackline Masters 190
Appendix 2: Some Free Resources 197
Appendix 3: Some Not-Free Resources 201
Bibliography 202
Index 207

Preface and Introduction

This book is mainly intended for preservice and inservice teachers of math at the PreK-8 levels, and parents and other caregivers of such students. The goal of this book is to help improve the informal and formal math education of PreK-8 students. The emphasis is on providing students with learning environments that help to increase their levels of math maturity. The learning environments stressed in this book include an emphasis on communication in the language of mathematics, the use of math-oriented games, and the use of math word problems.

The next paragraph is a short definition of a mathematically mature adult. The level of math maturity described comes from years of appropriate informal and formal education and math-related experiences. Later parts of the book will provide a more detailed definition of math maturity and more detail about possible roads leading to an increased level of math maturity.

Mathematically mature adults have the math knowledge, skills, attitudes, perseverance, and experience to be responsible adult citizens in dealing with the types of math-related situations, problems, and tasks that occur in the societies and cultures in which they live. In addition, a mathematically mature adult knows when and how to ask for and make appropriate use of help from other people, from books, and from tools such as computer systems and the Internet.

Scattered throughout this book you will find short **Math Maturity Food for Thought** subsections such as the one given below. Each asks you to reflect on a particular idea or presents you with some problems that you and/or your students might explore. Such reflection, introspection, and problem-solving challenges are an important aid to learning and to increasing one's level of math maturity. If you are using this book in a course, these subsections can be used in small group and/or large group discussions and sharing. This can be done in a face-to-face environment or via use of telecommunications systems.

Math Maturity Food for Thought. It's A-OK to have one's income taxes prepared by an expert or for a person to make use of income tax preparation software. The income tax system and tax law in the United States are frightfully complex and include substantial changes from year to year.

It is not possible for a person to gain and maintain a high level of personal expertise in every type of problem area that adults must routinely deal with. Thus**, knowing when and how to ask for math-related help (from a person or from a machine) —and how to make effective use of such help—is an important aspect of math maturity.**

Think about the math that you do for yourself in your everyday life, and the math that you do with the help of other people and/or with the help of calculators, computers, GPS, and so on. Do you consider yourself to be a mathematically mature adult? What could you do to increase your level of math maturity?

Math is a vast and steadily growing discipline. Moreover, math is an important component of science, technology, engineering, and many non-science disciplines. As an example, think about

the complexities involved in identifying, understanding, and attempting to deal with various aspects of global sustainability. These immensely difficult problems not only involve science, technology, engineering, and math (STEM), they also involve governments and politics, businesses and economies, and the lives of the people and other species on earth.

Common Core State Standards Initiative

In March 2010 the Common Core State Standards Initiative released a draft of its proposed standards, and this set of standards has been widely adopted. See http://www.corestandards.org/. Quoting from the proposed math standards (with bold face added to highlight emphasis on mathematical maturity):

> The draft Common Core State Standards for Mathematics endeavor to follow such a design, not only by stressing conceptual understanding of the key ideas, but also by continually returning to organizing principles such as place value or the laws of arithmetic to structure those ideas.
>
> The standards in this draft document define what students should understand and be able to do. Asking a student to understand something means asking a teacher to assess whether the student has understood it. But what does mathematical understanding look like? One hallmark of mathematical understanding is the ability to justify, in a way appropriate to the student's **mathematical maturity**, why a particular mathematical statement is true or where a mathematical rule comes from. There is a world of difference between the student who can summon a mnemonic device such as "FOIL" to expand a product such as $(a + b)(x + y)$ and a student who can explain where that mnemonic comes from and why it works. Teachers often observe this difference firsthand, even if large-scale assessments in the year 2010 often do not. The student who can explain the rule understands the mathematics, and may have a better chance to succeed at a less familiar task such as expanding $(a + b + c)(x + y)$. Mathematical understanding and procedural skill are equally important, and both are assessable using mathematical tasks of sufficient richness.
>
> The draft Common Core State Standards for Mathematics begin on the next page with eight Standards for Mathematical Practice. These are not a list of individual math topics, but rather a list of ways in which developing student-practitioners of mathematics increasingly ought to engage with those topics as they grow in **mathematical maturity** and expertise throughout the elementary, middle and high school years.

Dice and Other Math Manipulatives

If you are a PreK-8 teacher of math, the chances are that you have easy access to math manipulatives such as dice, spinners, small cubical blocks, pattern blocks, and so on. Some of these math manipulatives are available at home in board games such as Backgammon, Dungeons and Dragons, Monopoly, and Yahtzee.

This book is designed to be used with the types of relatively inexpensive math manipulatives available in schools.

Math Manipulatives Used in This Book

Here is a list of some of the manipulatives that are explored in this book. The book focuses on use of inexpensive manipulatives. See Appendix 2 for some suggested sales outlets.

- Base-10 blocks
- Coins (pennies, nickels, dimes, and quarters) or imitation coins.
- D6 (six-faced dice). Note that people often call these six-sided dice.
- D10 (ten-faced dice). Note that people often call these ten-sided dice.
- Dictionary (hardcopy or online).
- Double sixes dominoes.
- Double nines dominoes.
- Paper, pencil, eraser, scissors, etcetera.

For the moment, get yourself a pair of dice or just imagine in your mind a pair of dice. Here are some things to do and to think about. These provide an example of a few ideas explored in the book.

1. What are some similarities and differences between a "physical, real" pair of dice and a "mental model" of a pair of dice? For example, does your mental model of a pair of dice allow you to tell (see in your mind's eye) the spatial layout of the six different patterns of dots? How many dots are on the face opposite to the face containing two dots? Can you visualize rolling a pair of dice and seeing (in your mind's eye) the results? Is your mind able to mentally produce the randomness that comes from rolling a pair of physical dice?

Mental modeling is a key aspect of thinking and problem solving in every discipline, and it is quite important in math. An increasing ability to do math-related mental modeling is a sign of an increasing level of math maturity.

Rolling a die produces a random integer between 1 and 6 inclusive. However, rolling a pair of dice and adding up the total of the two dice does not produce a random integer in the range of 2 to 12 inclusive. Can you explain why, at a level that would be understandable to your peers or to children you work with? The ideas of random numbers and randomness are quite important in math and science. Thus, one measure of increasing math maturity is an increasing level of understanding of this topic.

2. Cut out 6 equal-sized small squares of paper. Write the numerals 1 to 6 on the six pieces of paper, one numeral on each piece. Then think carefully about whether putting these 6 pieces of paper in a box, carefully shaking or stirring them up, and drawing out one of them is mathematically equivalent to rolling a D6 (a six-faced die). Note that we now have the ideas of "physical, real" dice, a mental model of dice, and a "pieces of paper" model of dice. What are advantages and disadvantages of each of these three different representations? At what age might a typical child learn to deal effectively with these three different representations? An increasing level of ability to deal with different

but representations of math-related objects is a sign of increasing math maturity.

3. An ordinary die is a cube. Each of its six faces has a different set of dots (typically, colored indentations). The six different sets of dots represent the six numbers 1 to 6. The total of the dots on two opposite faces of a die is 7.

Why do you suppose that the faces are numbered so that the sum of the numbers on two opposite faces is 7? Is there some historical reason for this? Does this numbering scheme have any affect when dice are used to generate moves in a game? Do the differing numbers of indentations on the different faces slightly unbalance the die, so that some outcomes from rolling a die are more likely than others? When a fair D6 is rolled, each of the six possible outcomes is exactly equally likely. Increasing levels of knowledge, skill, and intrinsic motivation to pose such questions are signs of an increasing level of math maturity. Willingness and ability to use your brain and an information retrieval system such as the Web as an aid in answering such questions are signs of an increasing level of math maturity.

Math Maturity Food for Thought. Even quite simple ideas, such as a six-faced die, can lead to mathematically challenging questions. Think about possible ways to tell if a die is fair. How does this topic relate to math? Think about whether this topic would interest the students you work with. Similar questions can be asked about a coin that is being flipped. A fair coin is equally likely to produce heads and tails.

Learning Math

Keith Devlin's book, *The Math Gene* (2000) argues that human natural language capabilities provide the basis for learning math. His book provides explanations of how number sense, numerical ability, and algorithmic ability all come from linguistic ability. Thus, he argues, all humans with intact brains are quite capable of learning a great deal of mathematics.

Here is a fundamental, but perhaps somewhat strange way to think about oral communication. Think about a speaker's oral utterance as a word problem. The listener faces the task of trying to understand the utterance (the word problem) and take a suitable action based on this understanding.

From that point of view, a young child's life is full of word problems. Consider the situation of a parent saying to a child who is playing with several toy cars of different colors: "Please hand me a red car." The child is gaining practice in understanding a complex request. Notice that this is a more complex request than: "Please hand me a car."

As the child begins to speak, the child becomes a creator of word problems. A two-way conversation is an ongoing sequence of exchanging word problems that involves listeners needing to very quickly "solving" the problems being received and speakers very quickly "creating" word problems (new utterances). Consider the following conversation:

"Mommy, may I please have a cookie?"

"Yes, dear, after you put your toys away."

The child has an "I want a cookie." problem. The child has learned that one way to solve the problem is to make a polite request. The mother's response is relatively complex. She asks the child to carry out a particular action before the cookie is made available. In essence, the child is asked to deal with delayed gratification and to first solve the "putting toys away" problem. An

increasing level of ability to deal with delayed gratification is a sign of increasing of overall maturity. Being able to deal with delayed gratification is an important aspect of gaining an increased level of math maturity.

You can see that long before students start kindergarten, most have developed considerable ability to solve and to create word problems "on the fly" as they carry on a conversation. This takes a tremendous amount of intelligence. These first few years of informal education are very important to a child.

Math as a Language

Children vary considerably in how good they are at receiving and sending precise sets of directions. With appropriate instruction and practice, children can improve in this area. When the instructions and expected actions are related to math, then improvements are an indication of an increasing level of math maturity.

Human natural language-learning capabilities are so great that if a child is raised in a bilingual or a trilingual oral environment, the child will become bilingual or trilingual in oral communication. Moreover, think about children raised in a musical home environment. Music is a type of language, and music is innate to humans. Children raised in a musical home environment will learn a great deal of music before they reach school age.

Now, consider mathematics. Math can be considered as a type of language. It is a discipline-specific language developed by humans. Based on the research of Devlin (2000) and others, we know that a child with an intact brain has the capacity to learn a great deal of mathematics. The extent to which this learning occurs depends on the quality and extent of the informal and formal math education the child receives.

The mathematical richness of the environments that children are raised in vary considerably—probably much more than the linguistic environments. In any case, for most children the mathematical richness is poor relative to the linguistic environment.

Based on this line of reasoning, the premise of this book is that math education can be substantially improved by increasing the math richness of the life of a child both at the preschool level and continuing on through the informal and formal education as the child grows toward adulthood. The book focuses on:

1. **Communication in the language of math**—getting better at oral and written communication with understanding, and thinking in the language of math.
2. **Math problem solving,** with special emphasis on word problems.
3. **Math-oriented games**—using games that create problem-solving and communication environments.

In these approaches to increasing math maturity, there is a focus on precision of communication. The vocabulary, rules, and logic in a math-oriented game or a math-oriented word problem tend to be quite precise. Games and word problems help to create environments in which children of all ages can practice learning, gain skill in learning to learn, gain skill in developing and using strategies, and move toward increased math maturity.

An increasing level of math content knowledge and skills is an important aspect of increasing math maturity. The math content emphasized in this book is based on the work of the National Council of Teachers of Mathematics.

Organization of this Book

The first part of this book contains the Preface and Introduction that you are currently reading. This is followed by Chapter 1, which explores games that make use of both numbers and words. These games may get you started in using games with your students. They help lay a foundation for subsequent chapters that define math maturity and explore various related aspects of math.

Each of the first nine chapters includes activities for use with students and activities that might be used in a college-level course based on this book.

After that comes a sequence of chapters that explore a variety of games that can be used over quite a grade level range. Ideas about math maturity are integrated into these chapters.

The remainder of the book includes a chapter containing some summary and final remarks, an Appendix of links to free resources on the Web as well as some useful Blackline Master, an Appendix on commercially available materials, an extensive Bibliography, and an Index.

Authors of This Book

In total, the two authors of this book have authored and/or co-authored nearly 90 academic books as well as hundreds of articles. For details, see http://iae-pedia.org/David_Moursund and http://iae-pedia.org/Robert_Albrecht. While Moursund and Albrecht have been professional colleagues for over 30 years, this is their first book-writing collaboration.

David Moursund

Robert Albrecht

Chapter 1: WordsWorth Games

> "Play is the work of the child." (Multiple sources, including Friedrich Froebel 1782–1852 and Jean Piaget 1896–1980.)
>
> "When tools become toys, then work becomes play." (Bernie De Koven.)

Language and math are closely related. The brain of a healthy human infant has some innate knowledge of quantity and of language. Moreover, it has a tremendous potential to learn more in these two areas.

This chapter introduces a game named *WordsWorth*. The name comes from a combination of *words* and *worth*. In this game, words are given numerical values (a numerical worth), following the rules of the game. For example, a WordsWorth game might assign these numerical values to letters: a = 1, b = 2, c = 3, and so on to x = 24, y = 25, and z = 26. The WordsWorth of a word is the sum of the word's letter values.

WordsWorth is designed to help children increase their knowledge of both language and math. The game comes with many variations that make it useful over a very wide range of grade levels and student abilities, from grade 1, "up, up, and away" to the highest levels of learning. Note that the "rules of a game" in combination with playing the game (making allowable moves and taking allowable actions in the game) can be thought of as understanding and solving a word problem.

We introduce the WordsWorth game here because it captures the flavor of what this book is about—helping students increase their levels of math maturity through use of math-oriented games and math word problems. Later chapters go into games and word problems in much more detail.

Introduction

For a young child, memorizing the sounds for the letters of the alphabet and some counting words serve as evidence of the natural language learning capabilities of an intact human brain. Memorizing the sounds of the alphabet is like memorizing 26 nonsense sounds. The letters of the alphabet are arranged in a particular order (called alphabetical order), but there is no underlying theory for this order. Thus, memorizing them in a particular order is another indication of the capabilities of a human brain.

The sounds for counting numbers are also memorized in a particular order. However, the order one, two, three, … has the underlying logic of the increasing magnitude of the natural numbers.

Typically, young children memorize (learn) names of the first several natural numbers in the environment of counting a collection of objects. That is, meaning is associated with the words. Eventually, a still higher level of meaning—cardinal number; the number of objects in collection—is tied in with learning the natural number words.

Many children can say the letters of the alphabet in alphabetical order and can say the numbers 1 through 10 in numerical order before they begin kindergarten. Many can count the

number of objects in a small collection, by forming a one-to-one correspondence between the natural numbers and the objects, and stating that the number of objects in the collection is the last number produced by the counting process. In addition, most can do simple addition through a counting process. Sesame Street (http://www.sesamestreet.org/) has helped many millions of children gain and hone these skills.

Kindergartens vary on the emphasis they place on learning to read and to do math. Increasingly, however, kindergarten includes an emphasis on both reading and math.

In first grade, the really big push on reading and math begins. Our society has decided that learning to read with understanding and learning to do math with understanding are fundamental components of the education that we want all children to receive.

Of course, your authors know that *math* is a noun. However, from a teaching and learning point of view it is quite useful to also think of math as a verb. We want students to gain a relatively high level of knowledge and skill in both reading and mathing (doing math) . (You know that mathing is not currently a word in the English language. Try it out with your students and see if they like this new, made up word.)The terms literacy and numeracy are often used in talking about these two ideas.

Our current educational system has a strong propensity to divide the curriculum into a number of pieces and to teach as if these pieces are unrelated or only vaguely related. Thus, a first grade student might receive 90 minutes of instruction in language arts and 45 minutes of instruction in math each day, with little or no intertwingling between the two. Other disciplines, such as art, music, physical education, science and health, and social science may get blocks of time on a daily or weekly basis.

WordsWorth is a game that intertwingles words and numbers. Thus, it fits in with both the language arts curriculum and the math curriculum. It provides an easy way to increase the overlap between the two disciplines.

Arithmetic Calculations in WordsWorth and Other Games

The various WordsWorth games all involve doing "table lookup" (looking in a table of letters and their values to determine the value of a letter) and arithmetic. Students in the early grades might start with counting as their means of calculation, and then move on to mental math and paper and pencil math as their arithmetic skills improve. Teachers and parents can make a decision as to whether or when calculators or base 10 blocks are allowed.

Appendix 1 contains Blackline Masters that many students will find useful in doing Wordsworth calculations.

A Simple Version of WordsWorth

This form of WordsWorth is designed for children who can recognize and spell some words, and who can do simple addition by counting or other means.

Start with the first nine letters of the alphabet, *a* to *i*, and assign counting numbers to them as follows:

a is assigned the number 1. We write this as a = 1.

b is assigned the number 2. We write this as b = 2.

c is assigned the number 3. We write this as c = 3.

d is assigned the number 4. We write this as d = 4.

e is assigned the number 5. We write this as e = 5

f is assigned the number 6. We write this as f = 6.

g is assigned the number 7. We write this as g = 7.

h is assigned the number 8. We write this as h = 8.

i is assigned the number 9. We write this as i = 9.

We call the numbers assigned to letters **letter values**. The letter value of *a* is 1, the letter value of *b* is 2, the letter value of *c* is 3, and so on. Figure 1.1 is a table representation of this information.

Letter	Value (letter value)
a	1
b	2
c	3
d	4
e	5
f	6
g	7
h	8
i	9

Figure 1.1. Table of numerical values for letters a–i.

Upper-case letters have the same letter values as their lower-case counterparts. Figure 1.2 shows a different way of representing a table of letter values.

a = 1	b = 2	c = 3	d = 4	e = 5	f = 6	g = 7	h = 8	i = 9

A = 1	B = 2	C = 3	D = 4	E = 5	F = 6	G = 7	H = 8	I = 9

Figure 1.2. Table of numerical values for letters a through i and A through I.

Form some words using just the letters **a** through **i**. Two-letter examples include **ad**, **be**, **fa** (of do, re, mi, fa musical fame), and many more that you and your students can contrive. Three-letter examples include **age**, **bee**, and **dad**. Next, find the WordsWorths of the words—the sums of the numbers (the letter values) corresponding to the letters in the words.

The word **be** has letter values b = 2 and e = 5. The WordsWorth of **be** is 2 + 5, which is 7. We write: be = 7.

The word **fa** has letter values f = 6 and a = 1. The WordsWorth of **fa** is 6 + 1, which is 7. We write: fa = 7.

The word **age** has letter values a = 1, g = 7, and e = 5. The WordsWorth of age is 1 + 7 + 5, which is 13. We write age = 13.

The word **bed** has letter values b = 2, e = 5, d = 4. The WordsWorth of bed is 2 + 5 + 4, which is 11. We write bed = 11.

The word **bee** has letter values b = 2, e = 5, and e = 5. The WordsWorth of **baa** is 2 + 5 + 5, which is 12. We write: bee = 12.

The word **dad** has letter values d = 4, a = 1, and d = 4. The WordsWorth of dad is 4 + 1 + 4, which is 9. We write: dad = 9.

Now you have the essence of the **WordsWorth Plus 1 to 9** game. The term "Plus" in the name of the game indicates it is a WordsWorth game making use of addition. Later in this book we will briefly describe other versions of WordsWorth games. The 1 to 9 indicates the numbers used in the game, and also indicates the number of letters (9) used in the game.

Math Maturity Food for Thought. What do you think of the idea of using algebraic notation such as a = 1, b = 2, and age = 13 with children in the first grade? Present some arguments for and against doing this.

For older students one might use function notation. Thus, WordsWorth Plus of be = 7 might be expressed WWP(be) = 7 and WordsWorth Plus of age = 13 might be written WWP(age) = 13. Or, using lower case function names, wwp(be) = 7 and wwp(age) = 13. What are your thoughts on this?

Simplest Game: Calculate the WordsWorth (Numerical Value) of a Word

Perhaps the simplest version of WordsWorth—and a good starting point for beginners—is to give the children a list of words and have them calculate the WordsWorth of each word. This idea is illustrated in the examples given above.

First, however, here is a question for you. Does it occur to you to wonder how many words can be formed using just the first nine letters of the alphabet? Perhaps you find this to be an interesting "word" problem. Suppose a number of students worked together to find as many words as possible using only the letters a through i. Aha! How about dividing the class into small

teams and having rules such as a 15-minute time limit, no use of a dictionary, and no copying words from other teams.

After time has expired, teams exchange their results and check for accuracy, using dictionaries as necessary. A team's score is the total number of (supposed) words listed minus two times the number of supposed words that are not actually words.

There are many possible **strings** of letters that are not words. How can one tell if a string of letters is actually a word? You might think that this is easy—just look up the string of letters in a dictionary. But, what dictionary?

There are 9 x 9 (that is, 81) two-letter combinations that can be made from the first nine letters of the alphabet. With a little effort, one can look up each two-letter combination in a dictionary. See Figure 1.3. However, there are 9 x 9 x 9 = 729 three-letter combinations, 9 x 9 x 9 x 9 = 6,561 four-letter combinations, and so on.

	a	**b**	**c**	**d**	**e**	**f**	**g**	**h**	**i**
a	aa	ab		ad	ae		ag	ah	ai
b	ba	bb*							bi
c									
d									
e				ed		ef		eh	
f	fa								
g									
h	ha				he				hi
i				id		if			

* Not in the Scrabble dictionary.

Figure 1.3. Table of 2-letter words

Math Maturity Food for Thought. Think about the math that emerges from consideration of strings of letters. For example, if we want to find all possible two-letter words that can be constructed from our 26 letter alphabet, we need to check 26 x 26 = 676 strings. There are 17,576 three-letter strings, 456,079 four-letter strings, and 11,881,376 five-letter strings.

Wow! The number of five-letter strings is far more than the total number of words in the English language. Would reading and writing be easier if no word was longer than five letters? Why do you think we use so many words that are more than five letters long?

The *Oxford English Dictionary* lists more than a half million words, and new words are added from time to time. It is considered to be the accepted authority of whether a string of letters is a word. Can you imagine looking through this dictionary to find every word that is spelled using just the first nine letters of the alphabet?

The hard copy (that is, printed on paper) *Oxford English Dictionary* is large and expensive. For the purpose of playing WordsWorth at school or home, we need a smaller, less expensive, and more readily available dictionary. As one specifies the rules for a particular version of WordsWorth, one specifies the dictionary or dictionaries that are used to decide whether a string of letters is a word.

Probably you are familiar with the game named *Scrabble*$^{(R)}$. The *Official Scrabble*$^{(R)}$ *Players Dictionary* lists words that are allowed in that game. Such a dictionary certainly could be used in WordsWorth. In a school setting, it is desirable to have a classroom set of some particular dictionary, so all students follow the same rules as to what constitutes a word. The Website http://dictionary.com provides a free online dictionary.

The only 1-letter words in the English language are *a* and *I*. All 2-letter and 3-letter words in The *Official Scrabble*® *Players Dictionary* are available online.

- MidZine – Scrabble – All 2-Letter Words http://www.msoworld.com/mindzine/news/proprietary/scrabble/features/all2s.html.
- MidZine – Scrabble – All 3-Letter Words http://www.msoworld.com/mindzine/news/proprietary/scrabble/features/all3s.html.

With beginning readers you might play WordsWorth 1 to 9 by giving them a list of words that are spelled using only letters *a* through *i*—perhaps from the 2-digit and 3-digit Scrabble word lists at the Internet sites shown above—and have them calculate the WordsWorth of each word. You might then make the game a little more complex by giving them a list of words, only some of which are spelled using just the letters *a* through *i*. The rules of this game are to identify all of the words that are spelled just using letters *a* through *i* and then calculate the worth of each word.

After beginning readers have had just a little practice with WordsWorth 1 to 9, you may want to increase the complexity of the game. Keep in mind that adding just a few more letters substantially increases the number of words that can be spelled using the given set of letters and also substantially increases the arithmetic calculation challenge.

More Complex Versions of WordsWorth Plus

WordsWorth Plus 1 to 26 uses the entire alphabet and the addition of numbers in the range 1 to 26 inclusive. Students playing this game use the table of values given in Figure 1.4.

a = 1	b = 2	c = 3	d = 4	e = 5	f = 6	g = 7	h = 8	i = 9
j = 10	k = 11	l = 12	m = 13	n = 14	o = 15	p = 16	q = 17	r = 18
s = 19	t = 20	u = 21	v = 22	w = 23	x = 24	y = 25	z = 26	

Figure 1.4. Table of values of 26-character alphabet.

A teacher or parent may want to give students a list of words and tasks such as:

- Find the WordsWorth Plus of each word in the list of words.
- Arrange the words in alphabetical order.
- Arrange the words in WordsWorth Plus (numerical value) order.

- Find the word or words in the list that have the smallest WordsWorth Plus.
- Find the word or words in the list that have the largest WordsWorth Plus.
- Is there a word in the dictionary being used that is not in the list, but that has a smaller WordsWorth Plus than any word in the list? (One proves that the answer is **yes** by finding such a word.)
- Is there a word in your dictionary that is not in the list but has a larger WordsWorth Plus than any word in the list? (One proves that the answer is **yes** by finding such a word.)

Notice that the last two bulleted items are relatively challenging math and word-oriented word problems. To prove that the answer to one of these questions is **no** requires searching through the list of all words in the designated dictionary. This is a good task for a computer. One indicator of an increasing level of math maturity is an increasing level of insight into what math-related problems might best be solved by a computer.

Math Maturity Food for Thought. Young children tend to ask many questions. Asking questions (posing interesting and challenging problems to be solved) is a very important aspect of an increasing level of maturity. However, a teacher dealing with a roomful of students can easily be overwhelmed by a plethora of questions. A teacher is challenged by a dual task of encouraging questioning and moving the class forward in a content lesson plan.

A partial solution to this situation is to help your students learn to pose questions that are relevant to the curriculum but that can be asked and answered in small group discussions. Think of some ways that you might make use of WordsWorth in this endeavor.

Some Fun and Challenging Variations

It is easy to think of variations of WordsWorth Plus 1 to 26. For example:

- In a designated dictionary, what 2-letter word has the largest WordsWorth Plus? Is there more than one 2-letter word that has this value? What 2-letter word has the smallest WordsWorth Plus? Is there more than one 2-letter word that has this value? One can ask the same set of questions for 3-letter words, 4-letter words, and so on.

  Notice that when you present students with these types of questions, you are presenting them with a math word problem. Remember—a word problem can be presented using a combination of oral and written language, diagrams, examples, and so on. You are using both non-math words and math words to present specific versions of a game. The game itself has rules of what one is allowed to do. This is not unlike the "game" of math with its rules.

- Is it possible to find 26 words, with the value of the first being 1, the value of the second being 2, the value of the third being 3, and so on up to 26? Think about this for a minute. The only letter that has a value of 1 is the letter a. Fortunately, **a** is a word. The only possible letter strings having a value of 2 are **b** and **aa**. Is b a word? No! Is aa a word? Fortunately it is—it is a type of lava. Otherwise, the problem would have no solution. Now you are off to a good start.

  Can you think of a word having a value of 3? Think about your thinking (do metacognition) as you attack this word problem. It is easy to make a list of all possible letter strings that

might be a solution. They are c, ba, ab, and aaa. Then all you have to do is decide whether one or more of these possible solutions is actually a word.

Such an "exhaustive search" (examining every possible solution to see if it is actually a solution) is a useful approach in problem solving. If the listing and examination process can be computerized, then it may be possible to apply this guess and check (trial and error, exhaustive search) strategy to problems that have a huge number of possible solutions.

Problem solving is a key aspect of math. Many math problems (as well as problems in other disciplines) do not have solutions. From early on, children can be learning that some problems are not solvable. Also, many problems have more than one solution. Do you think that the problem presented here has more than one solution?

- What is the longest word (most letters) that has a given WordsWorth Plus? For example, what is the longest word that has a WordsWorth Plus equal to 18? Is it acacia, which has six letters? Can you find a word with more than six letters that has a WordsWorth equal to 18? What is the longest word that has a WordsWorth equal to the number of weeks in a year (52)? We know a word with 11 letters that has a WordsWorth Plus equal to 52. (Hint: It is a word sometimes spoken by magicians.)

Final Remarks

WordsWorth has a number of characteristics of a good educational game. Much of the value of a good educational game rests in the hands of the people who help children learn to play the game, and provide guidance to the learners that explicitly focuses on the important educational aspects of the game.

In many games, a beginner can learn the rules of the game from a novice player who is only slightly more advanced. However, neither the beginner nor the slightly more advanced novice can provide a focus on important educational aspects of the game. For example, one of the learning goals in using WordsWorth is to help students learn about tables, table lookup, and speed and accuracy in making use of tables. Think about what you know about uses of tables to represent data and to help solve problems both in math and in other disciplines. Certain functions can be represented by giving a table that contains the elements of the domain of the function and the value of the function for each of the domain elements.

What do you want students to learn about functions as they play a WordsWorth game? Remember, a function has a domain and a range. A function maps each element of the domain into exactly one element of the range. For the WordsWorth games, typically the domain is the set of words in a designated dictionary or word list. The range is some subset of the natural numbers.

A good game tends to have the characteristic of easy entry (it is easy to get started) and a high ceiling. For example, a beginner may master the rules and moves in a game such as checkers or chess in a modest period of time. However, it takes years of study and practice to get really good at playing these games. Such games provide a student with the opportunity to travel along the path from being an absolute novice to becoming more and more skilled at solving the types of problems inherent to the game.

Games such as WordsWorth Plus and its variations given in this chapter provide an opportunity for students to develop strategies that will help them play the game more efficiently and effectively. Of course, someone can tell a student a particular strategy and explicitly teach

the strategy. However, one of the goals in the use of games in education is to help students learn to develop strategies on their own, and then to explore possible uses of the strategies in problem solving outside of the realm or context in which the strategy was discovered and learned. A good teacher can be of great help in this learning task.

Activities and Possible Homework Assignments

This book is divided into three major sections. The Preface and Chapters 2–9 provide background information about math maturity. Chapters 1 and 10-14 present games, word problems, and math communication types of activities for use at the various PreK-8 grade levels. The remainder of the book contains a concluding chapter, three appendices, an extensive set of references, and an index.

Each chapter in the first section ends with a small set of activities designed to whet your appetite in working with children, and/or for use in class discussions and homework if you are using this book in a college course.

1. **(For use with students.)** Choose a simple version of WordsWorth Plus that you feel will be easy for most of the students in your class or for the children you are working with. Explain and illustrate how to play the game, and then have them play. After they have gained an initial level of experience, carry on a whole class discussion about what is fun, what is interesting, what is challenging, and so on in the game.

2. **(For use with students.)** Your students are quite likely used to the idea of the teacher presenting a type of problem in oral and/or oral and written form, and then showing a number of examples of how to solve the problem. The students are then asked to solve a set of such problems. This approach to math teaching ignores the goal of students learning to understand and make use of oral and/or written sets of rules or descriptions of a problem situation. As you teach math, hold in mind the very important goal of students learning to be effective receivers of oral and written communications that involve math.

   Select a difficulty level of WordsWorth Plus suited to children you work with. Present the game to them either orally or in written form. Do this in a manner that challenges their abilities to deal with an oral or written communication. The learning goal is for students to learn by reading and/or listening, rather than by imitation.

3. **(A possible homework assignment or discussion topic in a course.)** The idea of a function is one of the more important ideas in mathematics. A table such as a = 1, b = 2, c = 3, and so on up to i = 9 defines a function with domain consisting of the letters a, b, c, …, i and range consisting of the integers 1, 2, 3, …, 9. The game WordsWorth Plus 1 to 9 defines a function that has as domain all words in a specified dictionary or wordlist that can be formed using just the first nine letters of the alphabet and a range consisting of a set of positive integers, the WordsWorths of the words in the domain. In your opinion, at what grade level should children begin to learn about and make use of the terms function, domain, and range?

4. **(A possible homework assignment or discussion topic in a course.)** A dictionary, thesaurus, encyclopedia, and telephone book are arranged in alphabetical order. That ordering is a very useful aid to a person looking up information in hardcopy versions of books. However, if a person is using a search engine to seek information in an electronic version of one of these books, then the person doesn't need to know anything about alphabetical order. Give some arguments for still having children learn the letters of the alphabet in a particular order and for learning about alphabetical order even though computers tend to obviate the need for use of such ordering.
5. **(A possible homework assignment or discussion topic in a course.)** Think about some of the board games, card games, and other games you played as a child. Identify one that contributed significantly to your education. What aspects of the game made it educational? What might have made the game of increased and longer lasting educational value?

Chapter 2: Introduction to Math Maturity

> "God created the natural numbers. All the rest [of mathematics] is the work of man." (Leopold Kronecker; German mathematician; 1823-1891.)
>
> To understand mathematics means to be able to do mathematics. And what does it mean doing mathematics? In the first place it means to be able to solve mathematical problems. (George Polya; Hungarian mathematician; 1887–1985.)

Math has long been a required part of the school curriculum. This is because some math knowledge, skills, and ways of thinking are deemed important for all students.

We know that math is quite useful in helping to represent and solve problems in many different academic situations as well as in many situations people encounter at home, at work, and at play. We know that the overall field of mathematics is very large and it is still growing.

We also know that students taking math courses vary widely in:

1. How much math they have "covered" (had a reasonable opportunity to learn) in the math courses and informal math learning opportunities they have had.
2. How well they learn, understand, communicate in, and think using the oral and written language of math.
3. How well they can apply their knowledge and skills in a variety of math-related problem-solving situations.
4. How well and how long they retain the math they have learned.

These four topics all relate to math maturity. A child's progress in each of these four areas is an indicator of the child's growing level of math maturity.

This chapter provides some background information about math maturity. An understanding of this chapter is fundamental to understanding and making effective use of the math-related games and math-related word problems presented in later chapters.

Introduction to Brain Science

The brain of a newborn healthy child has a number of built-in capabilities and potentials. The term *plasticity* is often used to describe a brain's ability to change over time. Changes are produced through informal and formal education, training, and experiences, as well as through reaction to disease, injuries, and drugs. Poor nutrition can severely damage a brain.

It takes about 25 years or so for a person's brain to gain its full physical maturity. Even after reaching physical maturity, a healthy brain maintains considerable plasticity and ability to learn.

The brain of a healthy newborn child is naturally curious and creative. It has a great ability to learn. When you learn, the learning takes place in your brain and the rest of your body. Learning in a biochemical process, with changes occurring at a cellular level.

Through study and practice, considerable learning can be incorporated into your brain/body at a subconscious or reflex level. You are familiar with this in situations such as keyboarding using a computer keyboard, playing a musical instrument, dancing and sports, playing action video games, tying your shoelaces, and so on.

But, have you also thought about this in terms of learning oral communication? Your brain has learned to hear sequences of word sounds, and automatically translate them into meaning. Through instruction and practice in reading, your brain has learned to translate "squiggles" (writing) on a piece of paper into meaningful ideas. In addition, your brain has learned to automate the speaking and writing components of communication.

Now, consider the same general ideas, but apply them to communication in the language of mathematics. Very young infants have a little bit of innate ability to recognize small quantities, such as 1, 2, and 3. Recent research suggests we have some innate ability to deal with simple fractions such as 1/2 or 1/3. However, it takes many years of informal and formal education, training, and practice to understand and effectively deal with the language of mathematics at a level deemed appropriate in our current society.

These same ideas also hold for learning in any other discipline of study and practice. Your brain has tremendous versatility, plasticity, and ability to learn. It also has innate creativity. Probably you are familiar with the French mathematician René Descartes' (1596–1650) statement, "I think, therefore I am." A more modern version of this statement might be: "I think consciously and creatively, therefore I am."

Through education, training, and practice your brain can develop considerable math-related automaticity in oral and written communication in the language of math, and thinking in the language of math. The information to be learned can come from:

- sources internal to your body (including from your brain);
- sources outside your body via your senses.

The book you are currently reading is an example of an external source of information. As you think about what you are reading, you are drawing on information freshly stored in your brain as well as information you have stored in your brain over past years.

As an example of learning from internal sources of information, suppose that you are in a sensory deprivation tank. While in the tank, you can still think about information stored in your brain, you can learn by combining this information in new ways, and you can pose and solve problems. For another example, we now know that a lot of learning and unlearning occurs at a subconscious level while one is asleep. We also know that one's subconscious can work on a problem that has been brought into one's brain by previous learning and thinking efforts.

Once data comes into your brain, much of it is processed at a subconscious level. This processing builds on your current knowledge and skills. As an example, consider all of the data bombarding your senses as you walk along a busy street or in the woods. Unless you are paying very careful attention, most of the data coming in through your senses is ignored (filtered out) without ever impinging on your consciousness.

One of the goals in schooling is to help students get better at focusing their attention on the content to be learned. One of the characteristics of a good teacher is being able to attract and hold students' attention. One of the characteristics of a good student is being able to focus

attention on what is being taught. As a child's brain grows and matures, it increases in its capabilities to focus attention.

One of the characteristics of increasing "learning maturity" is increasing ability to focus one's attention on what is to be learned. Quoting Michael Posner, a world-class researcher on the topic of attention (Fermandez, 2008):

> … there is not one single "attention", but three separate functions of attention with three separate underlying brain networks: alerting, orienting, and executive attention.
>
> 1) Alerting: helps us maintain an Alert State. [To read and understand a sentence, you must be in an alert state.]
>
> 2) Orienting: focuses our senses on the information we want. For example, you are now listening to my voice. [You hear Mike Posner's voice through his writings.]
>
> 3) Executive Attention: regulates a variety of networks, such as emotional responses and sensory information. This is critical for most other skills, and clearly correlated with academic performance. … [This is why teachers spend so much effort trying to get students to pay attention.]
>
> The development of executive attention can be easily observed both by questionnaire and cognitive tasks after about age 3–4, when parents can identify the ability of their children to regulate their emotions and control their behavior in accord with social demands.

Math Maturity Food for Thought. You have undoubtedly heard of Attention Deficit Disorder (ADD) and Attention Deficit Hyperactive Disorder (ADHD), so you know that people vary considerably in how well they can focus their attention. The discussion given above suggests that our sensory systems and brain are designed to not pay conscious attention to most of the data coming in through our senses.

You have also heard of the idea of multitasking. In multitasking, one pays attention to two or more tasks at the same time. Analyze your strengths, weaknesses, and personal experiences in multitasking. Relate your insights to your own learning experiences and to your experiences as a teacher.

In summary, this section presents a somewhat simplistic model of learning that consists of three key ideas:

1. All of the learning you do occurs inside your brain and the rest of your body. Learning involves biochemical changes at the cellular level in your brain and in the rest of your body.
2. The data that is processed to produce learning can come from internal and external sources. In both cases, the learning that occurs is based on (constructed upon) what has been learned in the past. That is the essence of the learning theory called constructivism. For some information about

constructivism in math education, see the Math Forum site http://mathforum.org/mathed/constructivism.html.

3. Paying conscious, alert attention to the topics and ideas you are trying to learn, reflecting on them, and doing metacognition (thinking about your own thinking) on them can help direct the subconscious learning process. (Note, however, in kinesthetic learning of sports, dancing, and so on, conscious attention and careful thinking can often get in the way of developing the mind/body subconscious interactive coordination and automaticity that is needed to attain a high level of performance. The goal is a type of automatic, subconscious type of performance.)

Learning Without and With Understanding

You know about the idea of rote memory with little or no understanding. Such learning is a major component of learning by very young children. You also know the importance of learning with deep and long lasting understanding. As a child gains increased cognitive capabilities, the child gradually moves from rote memory learning with very little understanding toward learning with considerable understanding.

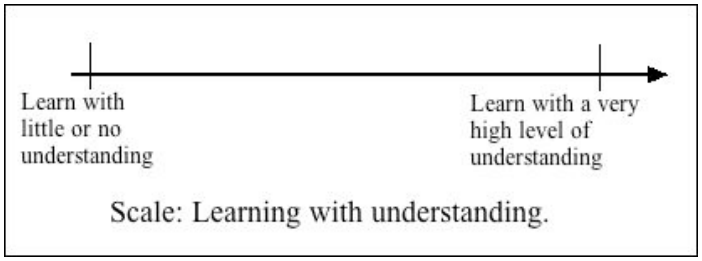


Figure 2.1 An increasing level of maturity in learning.

This progress toward learning with understanding can be thought of as progress toward greater cognitive maturity. It comes about through a combination of nature (genetic dispositions; your genetic blueprint) and nurture. Nurture includes things such as food, clothing, shelter, health care, protection against various dangers, informal education, formal education, and so on. Nurture also includes the loving and nurturing care that a parent and others can give to a child.

As adults, we sometimes tend to have little memory of the complex learning tasks we encountered and accomplished as children. Think about a child learning about the ideas of quantity, time, distance, length, area, volume, and so on. These are all very complex ideas.

To take a specific example, consider the challenge a child faces in learning to "tell" and to "understand" time. There is a huge difference between being to read a digital watch and say the numbers that represent the time, and having an understanding of what the numbers mean.

Think about helping a child to understand time. In the "good old days," all we had were analog clocks and watches. The passage of time was indicated by hands moving (rotating) around the center of the clock or watch face. The clock face typically contains only 12 numbers. A minute is the amount of time that it takes for the long skinny hand (the second hand) to make a complete rotation. One can see the second hand moving, and "watch" the passage of a minute.

Math Maturity Food for Thought. In the English language, one can talk about a first object or event, a second object or event, and so on. We talk about second hand goods, second hand information, and second hand smoke. In measuring time, we talk about seconds, minutes, and so on. We talk about the second hand on a clock. Hmm. We have used the word **second** in a variety of ways.

Confusing, right? And, of course, we have words that sound the same but are spelled differently and have different meanings. The words *four, fore,* and *for* are homonyms. A child learning to understand spoken English gradually comes to understand that the exact same sound pattern has different meanings in different contexts. The listener needs to recognize or figure out the context in order to assign correct meaning in homonym situations.

The language of mathematics is designed for very precise communication. However, the language of mathematics draws heavily on natural language. As an example, think about what the words *equal* or *equals* mean in English, and what they mean in math. Give some other examples of the math discipline assigning a quite specific and precise meaning to an English word, where the English word has two or more different meanings. Think about what this situation has to do with math maturity.

Digital timepieces are great things. But, an analog timepiece with rotating hands is a better aid to developing an initial understanding of time than is a digital timepiece. It is more concrete—think in terms of Piaget's concrete operations stage of a person's cognitive development.

Just for the fun of it, let's carry this time-telling learning for understanding a little further. What is a second, or a minute, or an hour? In your mind, compare and contrast this question with the ideas of a day, a month, and a year. A day (rotation of the earth on its axis), month (orbiting of the moon around the earth) and a year (orbiting of the earth around the sun) are words referring to physical, observable events. That is, they have concrete referents. Humans created the abstract concepts we call a second, a minute, and hour. If your head is not yet spinning, think about when and why we have leap years and why there is a difference between a lunar month and a calendar month. Time is a very complex topic!

To summarize this section, we all have some insights into learning with little or no understanding, and learning with understanding. We all have some insights into the range of complexity in the informal and formal education that children are exposed to. Some ideas, such as time, are so complex that they challenge the world's leading physicists. (Perhaps the name Albert Einstein just popped into your consciousness?) Thus, when we say that we want students to learn with understanding, we need to think carefully what level of understanding we are striving for.

Counting and Adding

Children learn counting words as such as one, two, three, and four as they are learning other worlds used in the everyday language environment in which they are raised. By the time children reach kindergarten, most can count the number of objects in a small collection. That is, they can make a one-to-one correspondence between the objects and the counting words, and then say that the number of objects is the last of the counting words that they used in making the correspondence.

This type of counting also allows a young child to add two numbers, such as two and four. Parents typically teach their children this relatively abstract concept. Perhaps a child has learned the following process to add two and four:

> Count (say the words) one, two. That takes care of the "two" part of the two numbers you are adding together. Then say the next counting word (three) as you bend down one finger. Say the next counting word (four) as you bend down a second finger. Say the next counting word (five) as you bend down a third finger. Say the word six as your bend down a fourth finger. You have now bent down four fingers, and this corresponds to the four in the addition problem. You are now all done. The answer is the last counting number (six) you named. (A slightly different approach is to start the finger bending with the one and two, and continue to bend fingers for the three, etc. The result is six bent fingers.)

A variation on this is to make use of small blocks. It is common to use base-10 blocks and other physical math manipulatives in elementary school. Counting on one's figures is such a powerful aid to doing simple arithmetic calculations that it carries on into adulthood for many people.

Think how complex this process is! Moreover, think about the transfer of learning that we expect of a child learning this process. We expect the child to recognize when the counting-to-add process is applicable. We expect the child to learn the algorithm well enough to apply it in situations not previously encountered.

A relatively powerful variation on counting-to-add is called *counting on*. To add three and seven, first select the larger number. Say its name (in this case, seven) and then begin the counting and finger bending process (saying eight, nine, and ten in this case).

Perhaps 2/3 of students discover on their own or are taught this counting on algorithm before they begin the first grade (Bruer, 1999). The correctness of this process depends on the assumption that the sum of two integers is the same regardless of the order of the two numbers. In math language, addition is commutative.

Math Maturity Food for Thought. Check your own level of understanding. Why is seven and three the same as three and seven? What sort of argument might be convincing to you or to a preschool child? Remember, this is a hard question. You "know" that three times five is the same as five times three. But, you also know that three divided by five is not the same as five divided by three. Thus, not all mathematical operations on pairs of integers are commutative. Can you think of any other math operations that can be carried out on pairs of integers, and that are not commutative? Think about what this situation means in terms of helping a child gain in math maturity.

The Number Line

Quite young children learn counting words such as one, two, three, … along with other frequently used natural language words in the languages they encounter. Children who grow up in bilingual or trilingual home environments readily develop bilingual or trilingual oral communication skills.

Children learn to count before they develop much of an understanding of the number line. In primary school they are apt to see a banner across the front of the room that is centered on 0 and shows a portion of the number line containing some of the positive integers and some of the negative integers.

You (as an adult) know that there are an infinite number of integers—so that the line number line extends "forever" in each direction. You know about the math symbol (infinity). You may have memorized that " is not a number." Ask yourself: if is not a number, what is it?

This provides us with another example of learning by rote memory with little understanding, versus learning in a more "mature" fashion with a level of understanding appropriate to your current and possibly future needs.

Let's explore the number line in a little more detail. The number line is a quite complex math idea. For example, it is "obvious" that there are exactly as many positive integers as there are negative integers. (You can form a one-to-one correspondence with +1 corresponding to -1, +2 corresponding to -2, and so on.)

But, consider the correspondence -1 corresponding to +1, -2 corresponding to +4, -3 corresponding to +9, and so on. In this one-to-one correspondence, each negative integer is paired with its square. That is, there is a one-to-one correspondence between the negative integers and the set of positive perfect squares. So, there are as many negative integers as there are positive perfect square integers. This example suggests some of the difficulties in understating .

You know, of course, that the number line contains more than just integers. For example, it contains fractions, such as1/2, 1/3, and 2/3. It contains other rational numbers that are not integers, such as 1 1/4, 5 1/3 and 24 2/3.

And, we could go still deeper. Consider a right triangle with its two shorter sides each being of length 1. The longer side—the hypotenuse— has a length that is not a rational number. The length is the positive square root of 2, and it is an irrational number. The number π (pi) is also an irrational number that you have encountered before. The discovery of rational and irrational numbers represented important events in the history of mathematics.

Math Maturity Food for Thought. Much of the math that students learn in our PreK-8 grades was known by the Greeks more than 2,000 years ago. Thus, for example, the Pythagorean Theorem is named after Pythagoras of Samos. Quoting from the Wikipedia: "Pythagoras the Samian, c. 570-c. 495 BC, was an Ionian Greek philosopher and founder of the religious movement called Pythagoreanism. He is often revered as a great mathematician, mystic and scientist; however some have questioned the scope of his contributions to mathematics and natural philosophy." Thus, when we have students memorize and work to understand the Pythagorean Theorem, we are teaching them a bit of history. However, we are failing to teach any of the important aspects of Pythagoras, his life, and his times. For example, what do you know about Euclid (of Euclidean Geometry fame)? What are your thoughts on this aspect of math education?

In brief summary, the number line is a very complex mathematical object. As one studies and learns about this complex object, there are different levels of learning and understanding. Thus,

when math teachers are told to "teach for understanding" and students are expected to "learn with understanding," things are not as simple as they might seem.

This situation leads to questions such as what constitutes an appropriate level of understanding for the number line that should be taught and learned in grade school, in middle school, in high school, or in college?

These types of questions help to complicate the topic of math maturity. Do we say that a student is increasing in math maturity as he or she learns more and more about the number line?

The same types of questions can be asked about any area of math. Certainly one aspect of increasing math maturity is gaining increased knowledge and skill in math. However, our focus on math maturity is not on specific math content. Rather, it is on math-oriented thinking, understanding, remembering, and being able to make effective use of the math that one has had an opportunity to learn.

Math has a High Inherent Level of Abstractness

Much of the power of math lies in its relatively high level of abstractness. Think again about a young child learning the number words one, two, three, etc. The child eventually learns that by saying the words and making a one-to-one correspondence with a set of objects, the final number said is the quantity of objects in the set. That is a major math-learning step.

Later the child encounters the symbols 1, 2, 3, etc. These are shorthand symbols for the words one, two, three, etc. and likely they are learned (memorized) before the child encounters and learns the alphabetic representations of the words one, two, three, etc. Do you find this interesting? We have children learn abstract shorthand representations for the natural language words one, two, three, etc. before we have them learn to read and spell the written forms of these words. That is, very early on in a child's education we move toward the abstractness and power of the language of mathematics.

David Tall is a mathematics education theorist at the University of Warwick in England. Quoting Tall (2000):

> The development of symbol sense throughout the curriculum therefore faces a number of major reconstructions which cause increasing difficulties to more and more students as they are faced with successive new ideas that require new coping mechanisms. For many it leads to the satisfying immediate short-term needs of passing examinations by rote-learning procedures. The students may therefore satisfy the requirements of the current course and the teacher of the course is seen to be successful. However, if the long-term development of rich cognitive units is not set in motion, short-term success may only lead to increasing cognitive load and potential long-term failure.

One thing implied here is that, as the symbols and the manipulations become more and more abstract, it becomes more difficult to relate to what is known; the student "learns" with less and less understanding. In many cases, a student would face a daunting task working out a referent that has meaning to the student.

Think about this in terms of a stage theory of cognitive development, such as Piaget's 4-stage model. Newborn children are in the sensorimotor stage. They gradually move into the preoperational stage and then the concrete operations stage. For most children, the concrete

operations stage roughly corresponds to the years of elementary school. In the concrete operations stage, the human brain is highly dependent on concrete referents. It struggles with the types of abstractness that are in the math curriculum. And so, many do not gain much understanding as they try to learn how to do arithmetic with fractions.

Piaget's fourth stage is called formal operations. At that level of cognitive development, people are able to deal with a high level of abstractness in math and in other disciplines. High school students in an Advanced Placement history course are expected to function at this level of cognitive development. See http://iae-pedia.org/Digital_Filing_Cabinet/Secondary_School_History.

We will discuss this more in **Chapter 4**. There we argue that the level of abstractness in the math curriculum tends to be quite high relatively to the cognitive development of students. This leads to memorization without much understanding—in order to pass the courses. As David Tall points out in the quote given above, this leads many students to failure (or, dropping out) in their math studies. The Algebra 1 course is a stumbling block for a great many students because they do not understand fractions and calculations involving fractions.

Defining Math Maturity

The Preface and Introduction to this book states:

> A **mathematically mature adult** has the math knowledge, skills, attitudes, perseverance, and experience to be a responsible adult citizen in dealing with the types of math-related situations, problems, and tasks that occur in the societies and cultures in which he or she lives. In addition, a mathematically mature adult knows when and how to ask for and make appropriate use of help from other people, from books, and from tools such as computer systems.

Chapter 1 and earlier parts of this chapter suggest a number of possible components of math maturity. Now, perhaps, you are expecting a definitive definition of what constitutes math maturity and how to measure it. If so, prepare yourself to be disappointed. There is no widely agreed upon definition of math maturity, and there are no assessment instruments that are effective in precisely measuring a person's level of math maturity.

However, there is quite a bit of literature on the topic. A recent Internet search of the expression *"math maturity" OR "mathematical maturity"* produced about 25,000 hits.

Quoting from the Wikipedia (http://en.wikipedia.org/wiki/Mathematical_maturity):

> Mathematical maturity is a loose term used by mathematicians that refers to a mixture of mathematical experience and insight that cannot be directly taught. Instead, it comes from repeated exposure to complex mathematical concepts.
>
> An illustrative example from common experience that may be more familiar to non-mathematicians would be high school geometry proofs. While most competent and interested students can follow a given proof and even determine whether or not it is correct, many still have trouble coming up with a proof. The ill-defined difference between those who can generate proofs and those who cannot is an example of a difference in mathematical maturity, in this case the ability to "see" how to proceed.

The book you are now reading disagrees with the part of the above definition that says, "that cannot be directly taught." The essence of this book is that informal and formal math education can be designed explicitly to help increase math maturity.

Many discussions of math maturity include a focus on understanding and creating proofs. Consider what this means for younger students. When a student solves a problem and then explains the steps involved in a manner that is convincing to others, the students has (in essence) created a proof. That is, problem solving and making proofs are two sides of the same coin.

George Polya was one of the leading mathematicians of the 20th century, and he wrote extensively about problem solving. *The Goals of Mathematical Education* (Polya, 1969) is a talk that he gave to a group of preservice and inservice math teachers. The talk focused on elementary school math.

> To understand mathematics means to be able to do mathematics. And what does it mean doing mathematics? In the first place it means to be able to solve mathematical problems. For the higher aims about which I am now talking are some general tactics of problems—to have the right attitude for problems and to be able to attack all kinds of problems, not only very simple problems, which can be solved with the skills of the primary school, but more complicated problems of engineering, physics and so on, which will be further developed in the high school. But the foundations should be started in the primary school. And so I think an essential point in the primary school is to introduce the children to the tactics of problem solving. Not to solve this or that kind of problem, not to make just long divisions or some such thing, but to develop a general attitude for the solution of problems.
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> …
>
> However, … We wish to develop all the resources of the growing child. And the part that mathematics plays is mostly about thinking. Mathematics is a good school of thinking. But what is thinking? The thinking that you can learn in mathematics is, for instance, to handle abstractions. Mathematics is about numbers. Numbers are an abstraction. When we solve a practical problem, then from this practical problem we must first make an abstract problem. Mathematics applies directly to abstractions. Some mathematics should enable a child at least to handle abstractions, to handle abstract structures.

Polya's comments focus on math problem solving, math thinking, dealing with math abstraction, and other aspects of "doing" math. In essence, the quoted material serves as a definition of math maturity.

Larry Denenberg has a PhD in applied mathematics and is a systems analyst, entrepreneur, and business executive. Quoting from a course syllabus developed by Denenberg (http://www.larry.denenberg.com/math22/LectureA.pdf):

> Thirty percent of mathematical maturity is fearlessness in the face of symbols: the ability to read and understand notation, to introduce clear and useful notation when appropriate (and not otherwise!), and a general facility of expression in the terse—but crisp and exact—language that mathematicians use to communicate ideas. Mathematics, like English, relies on a common understanding of definitions

and meanings. But in mathematics definitions and meanings are much more often attached to symbols, not to words, although words are used as well. Furthermore, the definitions are much more precise and unambiguous, and are not nearly as susceptible to modification through usage. You will never see a mathematical discussion without the use of notation!

The remainder of this section provides a list of some generally agreed on components of math maturity. This list, and components that you may want to add to fit your personal insights into math maturity, can help in the design of math curriculum content, instructional processes, and assessment.

The list given below is rather extensive.

1. Communication

Communicate mathematics and math ideas orally and in writing using standard notation, vocabulary, and acceptable style. As noted earlier in this book, an oral communication between two people can be thought of as an exchange of word problems. If the topic is related to and/or involves math in some way, then the two participants are involved in creating, communicating and perhaps solving math word problems.

Here is a more mathematical way of describing an increasing level of the communication component of math maturity. Consider a situation in which a person is using oral language, gestures, written languages, pictures, and diagrams, to communicate a math problem or math related problem to another person. The idea is to communicate the problem carefully and fully, so that the receiver of the communication can then bring his or her math knowledge and skills to bear in attempting to solve the problem. This type of communication requires a high level of precision on the parts of the two participants. An increasing level of math maturity is evidenced by increasing ability to communicate in math and math-related areas.

2. Learn to learn math and help others learn math

Learning math means learning with an appropriate combination of memorization and understanding. Some key ideas include constructivism, metacognition, and reflective thinking. Learning to learn math includes learning to make effective use of the various aids to learning that are available, such as teachers, peers, books, and computers. It also includes leaning to make effective use of one's overall learning knowledge and skills, and one's specific math learning knowledge and skills. An increasing level of math maturity is evidenced by increasing ability to be a self-sufficient intrinsically motivated learner who learns math with understanding. Increasing math maturity is also evidenced by increasing ability to work with people having varying levels of math knowledge and skills, and to help them learn math.

3. Generalize from a specific example to a broad concept

Mathematicians often start from a specific example of a problem, and go on to represent, define, and solve a broad category of closely related problems. For example, one might start with a concrete example of a problem involving a specific equilateral triangle, and develop results that solve this type of problem for every equilateral triangle. Increasing math maturity includes getting better at identifying a general class of problems from a specific example, in solving the general class of problems, and in making use of a solution to a general class or problems to solve specific instances of the problem.

4. Transfer of learning

One aspect of learning math is to learn a variety of strategies (algorithmic and heuristic) that are useful in attacking a broad range of math problems, and to learn to develop such strategies on one's own. Another aspect of learning math is to learn to think and reason mathematically. Increasing math maturity is evidenced by getting better at transferring or applying one's math knowledge and skills into other areas of math and into math related areas and problems in disciplines outside of mathematics. Progress is shown by increasingly being able to apply one's math knowledge and skills to challenging math-related problems and problem situations that one has not previously encountered.

5. Multiple, varied representations

Children begin their leaning of math well before they reach the "concrete operations" phase of cognitive development. This early math learning is rooted in verbal, tactile, and visual representations of specific concrete objects and events. Increasing math maturity is evidenced in increasing ability to deal with generalizations and less concrete examples. For example, there is a difference between working with three blue toy cars and two red toy cars that one has physically sitting before one's eyes, and doing the same thing with pictures in a book or pictures of cars in one's mind's eye. At a much higher math level, increasing math maturity includes getting better at moving back and forth between the visual (e.g., graphs, geometric representations) and the analytical e.g., (equations, functions) math representations.

6. Math problem solving and proofs

Solving math problems and proving math theorems lie at the very heart of mathematics. Increasing math maturity is evidenced by progress in being able to provide solid evidence (informal and formal arguments and proofs) of the correctness of one's efforts in solving math problems and making proofs. This is a specific type of communication in the language of math. Before students encounter formal math proof processes, math proof is often stated as "Show your work and check your answers." This means to present written explanations and arguments that support and assertion that your work is correct.

7. Math-related word problems

Represent (model) verbal and written problems in any discipline as mathematical problems. Recognize when a word problem might make effective use of math in attempts to solve the problem. Increasing math maturity is evidenced by increasing knowledge and skills in representing word problems using the language of mathematics, solving the resulting math problem, translating the results back into the language and context of the original word problem, and checking for accuracy and mindfulness of the math results in light of the context and meaning of the original word problem.

8. Math is a human endeavor

Math is more than just solving math problems and making math proofs. Our accumulated math knowledge represents considerable human creativity over thousands of years. Math is part of our culture. Math is fun. Math is part of the games we create and play. Math is part of the beauty of our world. (See Stephen Brown, n.d.) Increasing math maturity is evidenced by increased understanding or and participation in these various aspects of the overall discipline of mathematics.

Some related topics include mathematizing, thinking like a mathematician, and (math) problem posing (Stephen Brown 1997). An increasing ability to mathematize (see the math in a problem situation; pose math problems) is an indicator of an increasing level of math maturity.

9. Math content

As you think about the math maturity components listed above, notice that few specific math content topics are mentioned. One needs to know some math in order to be able to demonstrate increasing math maturity. But, increasing levels of math maturity are not dependent on gaining some specified and widely agreed on collection of math content.

An indication of an increasing level of math maturity is a student's active engagement in improving his or her math education and prowess in using this math education. This effort need not be focused on just the curriculum content being offered by one's school. Indeed the ability and interest to explore math-related ideas that happen to seem interesting is a good indicator of an increasing level of math maturity.

Great effort has gone into the development of various curricula and various approaches to learning math. There is a reasonable amount of agreement as to math topics that students should learn something about in elementary school, secondary school, and in undergraduate programs of study. However, it is depth of understanding that is the key idea.

Math is an old and vast discipline. It has great breadth and depth. It is a growing field in its own right, and its uses in other disciplines continue to grow.

The leaders and educators in each discipline design and construct curriculum in the discipline. They want students to gain some of the important knowledge and skills in the discipline and to lay a foundation for future learning within the discipline.

Thus, part of one's increasing math maturity is increasing breadth and depth of mathematical knowledge and skills. However, because of the great breadth and depth of the discipline of math, a mere listing of topics to be studied is a poor approach to improving math maturity. Please reread the words of George Polya quoted earlier in this chapter. In essence, Polya is stressing learning to think logically-mathematically in representing and attacking problems.

> To understand mathematics means to be able to do mathematics. And what does it mean doing mathematics? In the first place it means to be able to solve mathematical problems. **For the higher aims about which I am now talking are some general tactics of problems—to have the right attitude for problems and to be able to attack all kinds of problems, not only very simple problems, which can be solved with the skills of the primary school, but more complicated problems of engineering, physics and so on, which will be further developed in the high school.** But the foundations should be started in the primary school. And so I think an essential point in the primary school is to introduce the children to the tactics of problem solving. Not to solve this or that kind of problem, not to make just long divisions or some such thing, but to develop a general attitude for the solution of problems. [Bold added for emphasis.]

In essence, Polya is stressing the need to gain increased skill in representing and solving problems using the math that one has learned. This transfer of learning—moving from one's

math content knowledge to being able to effectively make use of the content in representing and solving problems—is a fundamental aspect of increasing math maturity.

10. Mathematical intuition

As one's knowledge of and experience in using math grows, one's math intuition grows. Herbert Simon, a Nobel Prize winning polymath, defined intuition as "frozen analysis." He noted that in any disciplines where one studies and practices extensively, a subconscious type of intuition is developed. This intuition may well be able to quickly detect an error that one has made in math thinking and math problem solving, very quickly decide a way to attack a particular type of problem, or provide a "feeling" for the possible correctness of a conjecture.

As an example, look at a student's statement that 5 + 8 = 40. At a subconscious level your brain might say, "something is wrong." It might next tell you, "the number 40 is way too large." Your experience and math teaching intuition might tell you, "perhaps the student multiplied instead of added." Through grading lots of student papers, you have developed some math intuition that makes you into a faster paper grader.

For a deeper view of math intuition, read Henri Poincaré 1905 paper, Intuition and Logic in Mathematics, available at http://www-history.mcs.st-and.ac.uk/Extras/Poincare_Intuition.html. Quoting the first paragraph:

> It is impossible to study the works of the great mathematicians, or even those of the lesser, without noticing and distinguishing two opposite tendencies, or rather two entirely different kinds of minds. The one sort are above all preoccupied with logic; to read their works, one is tempted to believe they have advanced only step by step, after the manner of a Vauban who pushes on his trenches against the place besieged, leaving nothing to chance. The other sort are guided by intuition and at the first stroke make quick but sometimes precarious conquests, like bold cavalrymen of the advance guard.

We see such intuition in other areas, such as in chess. An accomplished chess player can glance at a board position and have a "feeling" for the threats and opportunities that the position represents.

11. Computers and other math tools

All of the above needs to take into consideration the various tools that have been developed to aid in representing and solving math problems and problems in which math can be a useful aid to their solution.

Calculators and computers are powerful examples of such tools. These tools are useful both in representing and solving math problems and also in learning math. Moreover, computational mathematics is now one of the major subdivisions of the overall field of mathematics. Thus, increasing levels of math maturity are evidenced by increasing knowledge and skills in making effective use of Information and Communication Technology both as an aid to representing and solving math problems and as an aid to learning math. See http://iae-pedia.org/Two_Brains_Are_Better_Than_One.

Math Maturity Food for Thought. To summarize the list given above, a mathematically mature adult has the math and math tools knowledge, understanding, and skills to accomplish the math-related activities in his or her overall set of adult responsibilities and problem-solving challenges.

Here are three questions to ask yourself and/or to discuss with others:

1. How well does your current level of math maturity fit your current needs and your overall aspirations and plans for your future? Think about your understanding of math maturity and arguments to support your answer to the question.
2. How does your current level of math maturity compare with that of your peers and with other people you know?
3. What are some of the things you do that help increase the level of math maturity of students, your children, and other people you interact with?

WordsWorth Plus and Math Maturity

This book contains a number of math-oriented educational games and word problems designed to help students gain in math maturity. One of the goals in this book is to help you—the reader—become better at analyzing a game or word problem from the point of view of its contributions to helping students gain in math maturity.

Keep in mind that math maturity is not something that a student achieves or does not achieve. Rather, math maturity is an open-ended goal. Thus, math maturity is not a unit to be covered in a math course at some specified grade level. Rather, all math instruction should contain aspects designed to help students increase their current level of math maturity.

Now, consider the WordsWorth Plus game and some of its variations presented in Chapter 1. In working with students at some grade level or at some math maturity level, you make a decision to have the students use WordsWorth Plus. Your decision might be based on one or more thoughts such as:

- You feel that the combination of numbers and words in the WordsWorth Plus games will give your students useful insight into relationships among numbers and words, use of tables involving both numbers and words, practice in doing mental arithmetic, and practice in creating patterns of letters and determining if they are words.
- You feel your students need a break from the usual grind of the math curriculum you are using. However, you want this "break" to contribute to your students' overall cognitive growth.
- You want your students to be more actively and creatively engaged in math-related activities and word-related activities in a manner that combines the two.
- You are looking for academically respectable activities that can engage students with a wide range of academic backgrounds and potentials.
- You want to actively participate in game-based action research. See http://www.alliance.brown.edu/pubs/themes_ed/act_**research**.pdf.
- You trust the judgment of the authors David Moursund and Bob Albrecht well enough so that you are willing to experiment with a game that they recommend.

Notice that this type of thinking does not require you try to determine the "one best game" for use with your students. Rather, you think about games that are familiar to you and that you feel with be useful in your teaching. Be open to experimenting with new games that you have

been exposed to but have not yet tried out with your students. You think about specific educational goals you want to achieve by using a game. You make a decision to use a particular game based on your insights into your students as well as you understanding of underlying goals such as helping your students gain in math maturity and getting your students more actively engaged in math-related learning activities.

Final Remarks

This chapter gives a variety of approaches to defining math maturity. There is no simple definition of various levels of math maturity and no simple pathway that schools or an individual student can take to achieve specific levels of math maturity.

However, our educational system can help students to understand the concept of increasing levels of math maturity and moving in the direction of increasing math maturity. Our math education system can create curriculum content, instructional processes, and assessment systems that help students increase in math maturity. This book is designed to help in that endeavor.

Activities and Possible Homework Assignments

1. **(For use with students.)** Probably you remember the game, "Mother, may I?" See http://www.ehow.com/how_16079_play-mother-may.html. In this game a leader (typically called mother, but who may be a male or female child in the group) interacts with a group of children who begin arranged in a line facing "mother." Mother selects a child and says something like, "You may take three bunny hops forward." The child responds, "Mother, may I?" Upon receiving permission, the child then takes three bunny hops forward. Of course, large steps, small steps, jumps, and so on can all be used in this game.

   What does this have to do with math and math maturity? Here are a few answers:

   a. Children follow a set of instructions to line up in a certain way—in a straight line.

   b. The leader (mother) singles out a child and provides permission to make a certain movement. The movement is given in terms of a number and a type of movement.

   c. The child must precisely follow the rules of the game by saying, "Mother, may I?"

   d. Upon receiving permission, the child must then follow a set of instructions given in words and a number.

2. **(For use with students.)** Now, consider some variations on the "Mother, may I?" game. Suppose mother has a pair of dice—one red and one green. Mother is the leader of a game in which two other students participate. One student is designed as the "red die" player and the other as the "green die" player. Instead of doing steps of bunny hops, a player moves along a number line, starting at 0 and moving toward larger numbers. Mother rolls the dice. The red die player reads the red die (suppose, for example it is a 4) and says, "Mother, may I move four spaces forward?" Mother responds with "yes" if that is the correct number on the red die, and no if it is not. After red completes his or her move, green follows in a similar manner. The game proceeds for a

predetermined number of moves, such as 10 or until a player reaches or exceeds a predetermined target number such as 50

3. **(A possible homework assignment or discussion topic in a course.)** Examine the list of 11 possible components of math maturity given earlier in this chapter. Based on your knowledge of math teaching and learning at the precollege level, select two or three of the components that you feel our math education system does well at, and two or three that it does poorly at. Present arguments to support your choices.
4. **(A possible homework assignment or discussion topic in a course.)** Reflect on your own math maturity. Based on this, do three things:
    a. Suggest one or more components that you feel should be added to the 11-component-list given earlier in this chapter. Justify your recommendation(s).
    b. Select a person (such as a fellow teacher) that you know fairly well. Compare and contrast your level of math maturity and this person's level of math maturity using various items from the 11-item list given earlier in this chapter. Note that this activity can be done in small groups in a class setting.
    c. Make some suggestions to yourself on things you could do to increase your current level of math maturity.

Chapter 3: Introduction to Math Intelligence

> "Mathematics is a more powerful instrument of knowledge than any other that has been bequeathed to us by human agency." "Each problem that I solved became a rule which served afterwards to solve other problems." (René Descartes, French philosopher, mathematician, scientist, and writer; 1596–1650.)
>
> "My familiarity with various software programs is part of my intelligence if I have access to those tools." (David Perkins; Professor, Harvard Graduate School of Education.)

This chapter focuses on math intelligence and the next chapter focuses on math cognitive development. Taken together, these two chapters explore the general idea of increasing math intelligence and math cognitive development through appropriate instruction and practice. Such instruction and practice is an important aspect of helping a student gain an increased level of math maturity.

The basic ideas presented are that math intelligence is a component of overall intelligence and that math cognitive development is a component of overall cognitive development. These ideas provide a framework for working to improve one's level of math intelligence and maturity in using one's math intelligence. They also provide a framework for working to improve one's level of math cognitive development and maturity in using one's math cognitive development.

These ideas provide one way to approach the field of math maturity. The levels and types of math maturity that a person can achieve depend on the nature and nurture aspects of their math intelligence and math cognitive development.

Background on Innate Human Math Capabilities

People vary in their general intelligence. Moreover, some are far more gifted in mathematics than others. See a 6:28 minute video about Terence Tao, one of the world's leading mathematicians, at http://www.youtube.com/watch?v=p6ZUeQv2yFQ&NR=1. Thus, one of the points to consider when discussing math maturity is the extent to which math intelligence plays a role in how high up a math maturity scale a student might go.

The book *The Math Gene* (Devlin, 1999) presents an argument that the ability to learn to speak and understand a natural language such as English is a very strong indication that one can learn math. Thus, as a first approximation, one can argue that the average person can succeed in math about as well as they can succeed in learning oral and written communication.

In essence, Devin argues that a student's development of math knowledge and skills is mostly dependent on informal and formal education coming from parents, teachers, television, games, and so on. See (http://www.nctm.org/conferences/content.aspx?id=1270) for his opening keynote presentation at the 2004 NCTM Annual Conference.

Devlin's work helps us get at two major weaknesses in our current overall math education system.

1. A great many parents were not particularly successful in learning math, and typically they do not provide a "rich" math environment for their children.
2. The levels of math maturity, knowledge of math, math pedagogical content knowledge, and interest in math vary considerably among elementary school teachers.

The outcome from this widely varying home situation and widely varying school system is that some students get an excellent preschool and elementary school math education, and some get a relative poor start in learning math. Because of the vertical structure of math education, the latter group faces an uphill battle in attempting to meet the math requirements for graduation from high school.

As a consequence of the home and school situations just described, many children receive their preschool math education and their grade school education from adults who have only a modest level of math maturity. Remember, math content knowledge and math maturity are not the same thing. We are not claiming that the parents and teachers do not know the math content that they are helping children to learn.

Innate Math "Gifts"

Quoting from Moursund's book on talented and gifted education (2006a):

> **Talented**: having natural ability; gifted.
>
> **Gifted**: having natural talent or special ability; unusually able; talented.
>
> Sometimes an attempt is made to differentiate between the two terms talented and gifted. For example, giftedness may be defined by a score on a general IQ test, while talent may be defined by knowledge, skills, and performance in a specific discipline area such as art, math, music, science, or writing.
>
> A somewhat similar approach is to talk about ability versus attainment:
>
> Ability tests measure a person's potential, for instance to learn the skills needed for a new job or to cope with the demands of a training course. Ability tests are not the same thing as attainment tests .
>
> From this point of view, abilities are gifts, while talents or attainments are what one develops from his or her abilities.

In this chapter about intelligence and math intelligence, we are interested both in nature-given math giftedness (innate abilities) and nurture-developed math talents.

Research indicates that several-month-old human babies have innate ability to recognize small quantities, such as noticing that there is a difference between two of something and three of that thing. A variety of other animals have somewhat similar innate sense of quantity. This initial number sense can be viewed as an initial (innate) level of math maturity.

Recent research supports the idea that a human brain also has some innate ability to deal with fractions. See Todd Bentsen (2009). Quoting from the article:

> "Fractions are often considered a major stumbling block in math education," said Daniel Ansari, PhD, at the University of Western Ontario in Canada, an expert on numerical cognition in children and adults who was not affiliated with the study.

> (See: http://psychology.uwo.ca/faculty/ansari_res.htm.) "This new study challenges the notion that children must undergo a qualitative shift in order to understand fractions and use them in calculations. The findings instead suggest that fractions are built upon the system that is employed to represent basic numerical magnitude in the brain," Ansari said.

In summary, informal and formal math education and math experiences build on an initial innate level of math maturity.

Intelligence and Intelligence Quotient (IQ)

People have long been interested in intelligence. Here is a very old quote:

> Did you mean to say that one man may acquire a thing easily, another with difficulty; a little learning will lead the one to discover a great deal; whereas the other, after much study and application, no sooner learns than he forgets? (Plato, 4th century B.C.)

Substantial research supports the contention that students of higher Intelligence Quotient (IQ) learn faster and better than students of lower IQ. A teacher in a typical elementary school classroom may have one or two students who can learn twice as fast (and better) than "average" students and one or two who learn half as fast (and not as well) as the "average" students.

The "and better" and "not as well" parts of the previous paragraph are important in thinking both about math education in general, and in thinking about math maturity. Increasing levels of math maturity are highly dependent on achieving an increasing level of understanding, fluency, transferability, and so on of the math one has had a chance to learn.

Here is a little history on measuring IQ. Quoting from Wikipedia:

> The Stanford-Binet test started with [the 1904 work of] the French psychologist Alfred Binet, whom the French government commissioned with developing a method of identifying intellectually deficient children for their placement in special education programs. As Binet indicated, case studies might be more detailed and helpful, but the time required to test many people would be excessive.
>
> Later, Alfred Binet and physician Theodore Simon collaborated in studying mental retardation in French school children. Theodore Simon was a student of Binet's. Between 1905 and 1908, their research at a boys [sic] school, in Grange-aux-Belles, led to their developing the Binet-Simon tests; via increasingly difficult questions, the tests measured attention, memory, and verbal skill. Binet warned that such test scores should not be interpreted literally, because intelligence is plastic and that there was a margin of error inherent to the test.

Notice the initial measures of intelligence measured attention, memory, and verbal skill. An ADD or ADHD student might do poorly on such a test. A child growing up in a "rich" verbal environment will tend to score much better on such a test than a child who was seldom read to and had relatively few opportunities to practice meaningful, interactive conversation. Clearly, some of the measurements in an IQ test provide indicators of potential levels of achievement of math maturity.

Considerable research over the past century has supported the idea that IQ tests tend to measure a common factor called g, for general intelligence General intelligence (g) can be divided into two major components or factors: fluid intelligence (Gf) and crystallized intelligence (Gc).

The first common factor, Gf, represents a measurable outcome of the influence of biological factors on intellectual development whereas the second common factor, Gc, is considered the main manifestation of influence from education, experience, and acculturation.

In summary, a brain has innate ability called fluid intelligence and it gains in developed talents, called crystallized intelligence.

Nature and Nurture

It is important to understand that intelligence depends on a combination of nature and nurture. On average, intelligence increases considerably as a person's brain matures, and later in life it decreases as one grows old. It is the norming process in IQ that artificially makes it appear that one's intelligence is not changing over the years. This norming process does statistical transformation so that the mean IQ at each age level is 100, even though the actual intelligence of students increases substantially from year to year until full brain maturity is reached at age 25 or so.

How much of one's intelligence is due to nature and how much is due to nurture? This is a difficult question and researchers have produced varying answers. See http://wilderdom.com/personality/L4-1IntelligenceNatureVsNurture.html for a long list of nature and nurture factors that have been studied.

Here is a somewhat different way of looking at this question. A newborn with a healthy brain has a tremendous capacity to learn. The child's brain grows rapidly and learns rapidly. Just imagine the challenge of gaining oral fluency in one language. If the child happens to live in a bilingual or trilingual home and extended environment, the typical child will become bilingual or trilingual. Amazing! This represents a huge capacity to learn and to make use of one's learning. The point is that the "average" person is very intelligent.

Studies of nature versus nurture typically make use of identical twins that were separated at birth. Findings vary—with indications of nature determining from about 50 percent to about 80 percent of IQ, depending on the particular study.

Current research suggests that nature and nurture work together in a very complex manner, and that we have a long way to go in this area of research.

However, we are making good progress on the nurturing side. Schools are designed to be nurturing. But remember, decreasing various poisons in our environment is a nurturing activity, and it can lead to increased intelligence. Thus, unleaded gasoline was a major step forward. Providing children with appropriate food (versus starvation diets) preserves the intelligence that nature endowed.

Multiple Intelligences

The human brain's complexity and ability to create new neural pathways and neurons mean that all normal people can learn and can deal with complex, challenging problems. Still, there are significant differences in mental abilities among people who have the same overall intelligence. It's easy to observe that some people have more linguistic ability, musical, or math ability than

other people. It's easy to conclude that a given healthy brain has significant built-in inherent abilities to learn better in some disciplines than others. To take an extreme example, some people's hearing systems have a unique ability to acquire perfect pitch, or nearly perfect pitch. That is a distinct advantage in achieving a high level of success in musical performance or composition.

Examples demonstrate that genetics can endow a person with genius-level mathematical-logic sense. At age three, Carl Friedrich Gauss saw a paper on which his father was doing complicated calculations. He noted an error and worked out the correction mentally, and passed on the information—all within a few seconds.

Wolfgang Amadeus Mozart composed some relatively good music by the time he was eight years old. However, music experts consider his early works as quite immature.

A multitude of such examples exist. This has lead to the study of multiple intelligences—the idea that a person has several different kinds of intelligence or varying levels of innate ability in various intellectual types of activities.

Howard Gardner and Robert Sternberg

The research and writings of Howard Gardner (see http://iae-pedia.org/Howard_Gardner) and Robert Sternberg (see http://en.wikipedia.org/wiki/Triarchic_theory_of_intelligence) provide us with two models of multiple intelligences.

Howard Gardner's theory currently includes nine different areas or types of intelligence:. See http://skyview.vansd.org/lschmidt/Projects/The%20Nine%20Types%20of%20Intelligence.htm.

1. linguistic,
2. logic-mathematical,
3. musical,
4. spatial,
5. bodily kinesthetic
6. interpersonal,
7. intrapersonal,
8. naturalistic, and
9. existential.

One way to think about this is in terms of a bell-shaped curve of intelligence levels for each of the nine areas. A person might be at a different point on such curves for each of the nine areas or types of intelligence.

Robert Sternberg's triarchic theory of intelligence contains three different areas of types of intelligence:

1. creativity,
2. analytic (sometimes referred to as school smarts), and
3. practical (sometimes referred to as street smarts).

From a math maturity point of view, one person might be much more math-related creative than another, and such creativity may be increased by study and practice. In terms of school smarts versus street smarts, Gene Maier's writings about Folk Math (Maier, 1976) provide some interesting insights. A person tends to develop an increased level of math maturity that serves the person in their life environment. The cited article gives examples of people who develop the "street smarts math" that they need to function well in their societies.

Thus, as we help students to increase their level of math maturity, we need to think about what the students are interested in and the nature and extent of the math that they will find useful in their adult lives.

Math Maturity Food for Thought. We know that with appropriate instruction and practice an average child can learn to read with understanding and to read well enough to learn by reading. This attests to the intelligence of an average human brain.

We also know that for a variety of reasons, quite a few students have trouble learning to read well. Our educational system works to identify such students relatively early on in their reading studies and puts considerable extra resources into helping these students.

While many students are able to learn math through the types of math curriculum available in our schools, many also flounder. A good indication of this floundering is not learning for understanding and not being able to adequately use the math that has been covered in the curriculum.

Analyze the reading versus math analogy. Do you feel that the great majority of students can learn math with understanding and gain an ability to use this math to represent and solve problems and as a foundation for learning more math? What evidence do you have to support your conclusions?

The Flynn Effect

IQ tests were widely used during World War I. Quoting from http://users.ipfw.edu/abbott/120/IntelligenceTests.html:

> During World War I, the U. S. Army saw a need for a quick-to-administer intelligence test to be used when deciding what sort of advanced training a recruit would receive. Psychologists Lewis Termin, Robert Yerkes, and others collaborated to develop two versions of the test, known as the Army Alpha and Army Beta tests. The Alpha test emphasized verbal abilities and was given to everyone. The Beta test emphasized non-verbal abilities and was to be given to those who performed poorly on the Alpha test and were suspected of having language problems.
>
> A large number of Army recruits took the Alpha version of the test. After the war, the data were analyzed and yielded a surprising result. It appeared that the average recruit had a mental age of around 13—a mild level of retardation. The reason for this had to do mainly with the level of education of the recruits rather than low native intelligence, but Yerkes and others concluded incorrectly that the intelligence deficit was real, sounding alarm bells about the "menace of the feeble-minded."

You may find it interesting and amusing to see some sample questions from the World War I intelligence tests. See http://historymatters.gmu.edu/d/5293. We have come a long way in testing since that time.

Huge numbers of people from throughout the world have taken intelligence tests. In the past decade, some researchers have done longitudinal studies of scores from these various tests. They have produced considerable evident that the average IQ has been going up at a significant rate. This has been called the Flynn effect. Quoting from the Wikipedia (see http://en.wikipedia.org/wiki/Flynn_effect):

> The Flynn effect is the rise of the average intelligence quotient (IQ) test scores over generations (IQ gains over time). The effect has also been reported for other cognitions such as semantic and episodic memory. The effect occurs in most parts of the world although at greatly varying rates.
>
> The effect's increase has been continuous and roughly linear from the earliest days of testing to the present. "Test scores are certainly going up all over the world, but whether intelligence itself has risen remains controversial," psychologist Ulric Neisser wrote in an article in 1997 in The American Scientist.

Quoting from Graham and Plucker (2002):

> Research shows that IQ gains have been mixed for different countries. In general, countries have seen generational increases between 5 and 25 points. The largest gains appear to occur on tests that measure fluid intelligence (Gf) rather than crystallized intelligence (Gc).
>
> **Fluid Intelligence**
>
> Tests like the Ravens, the Norwegian matrices, the Belgian Shapes test, the Jenkins test, and the Horn test are examples of tests that attempt to measure fluid intelligence. These tests try to emphasize problem solving and minimize a reliance on specific skills or familiarity with words and symbols. These tests on average have shown an increase of about 15 points or one standard deviation per generation (Flynn, 1994); Flynn, 1987). Deary (2001) notes that it is these types of tests (i.e., "culturally reduced") on which we would not expect to see score increases if the cause of the increases was due to educational factors.
>
> **Crystallized Intelligence**
>
> Tests like the Wechsler-Binet and purely verbal tests measure crystallized intelligence in addition to fluid intelligence. Some questions on these tests measure problem-solving abilities but others measure learned information such as vocabulary and math skills. The IQ gains for these tests have been more moderate, with an average of about 9 points per generation (Flynn, 1994; Flynn, 1987). [Note: A "generation" is not a precisely defined length of time. In the quoted material here, the word typically means about 25 to 30 years.]

This type of research provides evidence that both fluid intelligence (due to nature) and crystallized intelligence (due to nurture) can increase in large populations. The increase has been so large that it has required renorming of IQ tests in order to maintain the average score at 100.

Math Maturity Food for Thought. Think about the Flynn effect and your own insights into our educational system. What evidence and/or examples are you aware of that suggest an average increase in the IQ of students is leading to a significant change in curriculum content and standards of performance that are being expected of students?

Outsmarting IQ: A Book by David Perkins

David Perkins has a rather unusual background for an education professor. His 1970 doctorate from MIT is in mathematics and artificial intelligence (AI). Now, he is a senior professor of education at the Harvard Graduate School of Education.

It is easy to see how math and AI fit together. Back in the 1960s, as computer science was beginning to be a relatively large and important discipline, the roots of computer science (CS) were in math, engineering, philosophy, and business. AI draws heavily from general aspects of CS and math. In more recent times, it also draws heavily from psychology and brain research.

While Perkins was doing his doctoral work, he became one of the founding members of Project Zero (n.d.) at Harvard. Project Zero is a long-running research project focusing on understanding and enhancing learning, thinking, and creativity in the arts, as well as humanistic and scientific disciplines. Eventually, after he became a faculty member at Harvard, he and Howard Gardner were the co-directors of Project Zero. This collaboration continued for more than 25 years.

In 1995 Perkins published a book titled: *Outsmarting IQ: The emerging science of learnable intelligence.* (Perkins, 1995). This book provides a very good introduction to the field of intelligence. It provides a good summary of the research on the design and implementation of curriculum that leads to increases in student IQ.

Artificial Intelligence

Artificial Intelligence (AI) is one of the subdivisions of the discipline of Computer and Information Science. Quoting from Moursund (2006):

> Artificial intelligence is a branch of the field of computer and information science. It focuses on developing hardware and software systems that solve problems and accomplish tasks that—if accomplished by humans—would be considered a display of intelligence. The field of AI includes studying and developing machines such as robots, automatic pilots for airplanes and space ships, and "smart" military weapons
>
> The theory and practice of AI is leading to the development of a wide range of artificially intelligent tools. These tools, sometimes working under the guidance of a human and sometimes without external guidance, are able to solve or help solve a steadily increasing range of problems. Over the past 50 years, AI has produced a number of results that are important to students, teachers, our overall educational system, and to our society.

Progress in AI is creating a major challenge to educators—especially those in fields where intelligent computer systems and robots can solve or help to solve many of the problems that are currently studied in school. As a very simple example, an inexpensive solar cell powered handheld calculator has the "intelligence" to add, subtract, multiply, divide, and calculate square

roots. A more expensive calculator or computer system can contain the software called a Computer Algebra System. See http://en.wikipedia.org/wiki/Computer_algebra_system. This calculator or computer can solve a quite range of the problems that students study in their math courses up through first year calculus and linear algebra courses.

One approach to this educational system is to embrace the ideas discussed in Moursund's (2010) discussion about *Two brains are better than one.* The argument is presented that human brains and computer brains each bring unique and valuable approaches to problem solving, and that students should be learning to make effective use of both types of brains. Also see the free Information Age Education Newsletter starting with issue # 31 available at http://iae-pedia.org/IAE_Newsletter.

Use It or Lose It

Brain research is now making very rapid progress due to the development of various brain scanning devices (Ratey and Hagerman 2008). Two really important findings are:

1. The brain has great plasticity. Thus, neurons can be repurposed. As an example, we now have computer game-like software that can be used to "cure" dyslexia and some forms of severe speech delay.
2. Use it or lose it. Both physical and mental exercise help to keep one's brain healthy.

For more information see http://iae-pedia.org/Brain_Science. This site also contains links to some excellent and free videos on brain science. Also see Posit Science at http://www.positscience.com/. This site contains links to recent research on brain exercises.

Your brain is active all of the time, even when you are asleep. Here is an analogy that you may find useful. Your heart and lungs function all of the time, even when you are asleep. However, through appropriate physical exercise, your heart and lungs can become more physically fit. Similarly, through appropriate cognitive exercises, your brain becomes more mentally fit.

Math Maturity Food for Thought. Think about the type of mental exercise involved in playing a game such as WordsWorth Plus versus the type of mental exercise involved in memorization of a long poem or a book.

Rote memorization is an important aspect of learning and one can get better at it by learning various memorization tricks and by practice. With great diligence, an average human can memorize a several hundred page book nearly letter perfect. This type of memorization can be done with little of no understanding of what is being memorized.

The above paragraphs raise the issue of what types of brain exercises we want to make use of in schools. Historically, students in the United States were expected to learn Greek and Latin before entrance into a college or university. What are your thoughts on brain exercise as a way to think about content and processes of education? For example, do you think that our current education system isn't providing students with enough mental exercise or the right types of mental exercise?

In recent years, educational leaders have begun to increase the emphasis on learning for understanding, routinely using this understanding, and for having students deal with cognitively

challenging problems. In math education, for example, this includes an increasing emphasis on depth of understanding rather that gaining a superficial rote memory type of breadth of math knowledge and skills.

Brainteasers

A brainteaser is a popular type of word problem. In this section, we include riddles, paradoxes, logic puzzles, and so on as brainteasers. Paraphrasing from the Wikipedia:

> A brainteaser is a form of puzzle that involves thinking activity to solve. Normally, this includes thinking in conventional ways with given constraints in mind; sometimes, it also involves lateral thinking (thinking outside the box). Logic puzzles and riddles are specific types of brainteasers.

Some brainteasers are math oriented and/or logic oriented. Here are some examples:

1. A square medieval castle on a square island is under siege. All around the castle there is a square moat 10 meters wide, very deep, and populated by monsters. Due to a regrettable miscalculation, the raiders have brought footbridges that are only 9.5 meters long. The invaders cannot abandon their campaign and return empty-handed. How can the assailants resolve their predicament?
2. You have a piece of paper that is exactly 20cm by 20cm. It has a total area of 400 square centimeters. For some reason, you need a square piece of paper with an area of exactly half this size. You have a pair of scissors, but you don't have a measuring device or a writing device such as a pen or pencil. How can you cut out a square with an area of 200 square centimeters from your larger piece of paper?
3. You are at a river that you want to cross with all of your goods. Your goods consist of a chicken, a bag of grain, and your large dog named Wolf. You have to cross the river in your canoe but can only take one passenger (chicken, dog, bag of grain) with you at a time. You can't leave the chicken alone with the grain, as the chicken will eat the grain. You can't leave your dog Wolf alone with the chicken, as Wolf will eat the chicken. However, you know that Wolf does not eat grain. How do you get everything across the river intact? Spend some time trying to solve this problem before looking at a solution given next.

Many brainteasers require thinking outside the box. As an example, here is a solution to the third of the brainteasers given above.

> **Solution:** Take the chicken across the river first and leave it on the other side. Return to where you have left Wolf and the grain.
>
> Next, take Wolf across the river, and leave him there, but bring the chicken back with you.
>
> Next, leave the chicken where you started. Take the bag of grain across the river and leave it with Wolf.
>
> Finally, go back and get the chicken, and take it with you across the river.

Many people do not think about the idea that in solving this puzzle you might bring something back on a return trip. They never consider this possibility, and they are unable to solve the puzzle problem. Note also that the learning gained from solving this particular puzzle probably will help you solve similar puzzles.

Here is a think outside the box brainteaser that you may have encountered before.

> Using pencil and paper, arrange nine distinct dots into a three by three pattern as illustrated in Figure 3.1. The task is to draw four straight line segments with the beginning of the second starting at the end of the first, the beginning of the third starting at the end of the second, and the beginning of the fourth starting at the end of the third, and so that the total sequence of line segments passes through each dot.

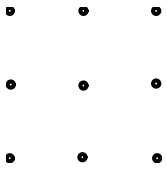


Figure 3.1. Nine dots in a 3x3 square pattern.

See if you can solve this puzzle before reading further.

To begin, you may think about how easy it is to complete the task using five line segments. A solution is given in Figure 3.2. After studying this solution, you can easily find other 5-line line segment solutions.

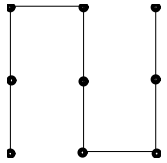


Figure 3.2. A 5-line segment solution for the 9-dots puzzle.

How can one possible complete the task with only four line segments? As with the river-crossing puzzle, it is necessary to think outside of the box. In this case, the layout of the puzzle tends to create a visual box. Many people do not think about drawing line segments that go outside of the visual box. A solution using four line segments is shown in Figure 3.3.

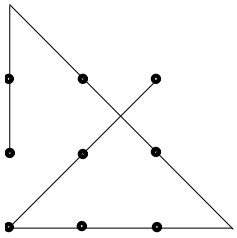


Figure 1.3. A 4-line segment solution for the 9-dots puzzle.

Many students enjoy working on brainteasers. You might want to collect brainteasers that are suitable for students you work with. Perhaps post a new one on the bulletin board each day or each week. There are lots of Web brainteaser resources that can easily be located by searching using expressions such as *math brain teaser* or *logic puzzle*. You might pick a couple of different students each week to be responsible for finding and posting the brainteasers for the week.

Final Remarks

People vary considerable in their innate math gifts. The quality and quality of their informal and formal math education experiences vary considerable. Moreover, people have quite varying levels of interest in the discipline of mathematics. Thus, nature, nurture, and levels of interest combine to make for huge differences in the math knowledge, skills, and levels of math maturity of different people.

Partly because of the vertical structure of math education curriculum, this situation places considerable burdens both on teachers and on their students. The math differences among students in a typical first grade class are large, and they increase substantially as students progress up through the grades.

One of the results is even through most students are innately capable of learning a great deal of math, many students come to look at math as an area in which they do rather poorly and that they don't like very well.

Activities and Possible Homework Assignments

1. **(For use with students.)** There are many different brainteaser problems. Often they are stated in words, and so they are examples of word problems. Try some of the brainteaser problems given above with your students. Discuss with your students what is fun, what is not fun, and what they learn by trying to solve such problems.

2. **(For use with students.)** Here is a type of variation on the "Mother, may I?" game. It is called the Robot game. One child in the classroom is designated to be the robot, and a second child is designated as the robot commander. The robot does exactly what the robot commander says.

   So, for example, suppose the robot commander wants the robot to pick up a book on the robots desk, carry it to the teacher's desk, set it down on the teacher's desk, and then return to and be seated at the robots desk. Before the activity begins, an agreed upon set of commands is developed. These might include things like: pick up the book that is in front of you; stand up; turn right one quarter turn, take one step forward, (or, take n steps forward where n is a specified number), stop, and so on. There is a precise set of instructions that the robot understands.

   The instructions that the robot understands may well include various math words. The instruction giver must make decisions that prevent damage to the robot (Don't run the robot into the wall!) Both the robot and the robot commander gain practice in precise communication. The robot commander gains practice in developing a sequence of instructions to solve a particular problem.

   Note also that this activity can lead to students wanting to learn more about capabilities and limitations of "real" robots. For example, a real robot might have a vision and distance-measuring system that it can use to avoid running into or over an obstacle. Students can make up sets of instructions that they think a robot should be able to understand and carry out. Students can create a program (a set of instructions) to be carried out by a robot in a manner that will accomplish a specified task. Other students in the class can mentally play being the robot, mentally go through the steps to accomplish a specified task, and determine if the

instructions are correct. As you can see, all of these types of activities contribute to building math maturity.

3. **(A possible homework assignment or discussion topic in a course.)** What are some similarities and differences between human brain and the "brain" of an inexpensive handheld calculator? In this discussion, focus on relative strengths and weaknesses, and situations in which a person using the two kinds of brains together can out perform a person using just their human brain.

4. **(A possible homework assignment or discussion topic in a course.)** Research on the Flynn effect suggests that over the years, people are getting more intelligence (as measured by IQ tests). One might conjecture that this increasing intelligence would manifest itself in students getting better in math, especially in elementary school where math is a required subject and students throughout the country cover much the same curriculum. What is your opinion as to whether our educational system is doing better in math education at the elementary school level, and what evidence do you have to support your opinion? (In dealing with this question, it is certainly all right to make use of reference materials.)

Chapter 4: Introduction to Math Cognitive Development

> "It is with children that we have the best chance of studying the development of logical knowledge, mathematical knowledge, physical knowledge, and so forth." (Jean Piaget; Swiss philosopher and natural scientist, well known for his work studying children and his 4-stage theory of cognitive development; 1896–1980.)

The field of cognitive development provides an excellent vehicle for exploring math maturity. As a child's brain grows and matures, the brain is increasingly able to deal with abstraction in math and in other disciplines. Research indicates that it is possible to enhance cognitive development. Thus, math education should be designed to "stretch" a learner's current level of math and overall cognitive development.

Piagetian Stage Theory

Jean Piaget is well known for the four-level (four-stage) cognitive development model that he developed. According to Piaget, a child's growing and maturing brain moves through four stages of cognitive development. These are not *abrupt* stages (there is gradual movement from one stage to the next). Moreover, one can also look at progress within specific disciplines—a student might move more or less rapidly in the math component of a stage. (See http://www.comnet.ca/~pballan/Piaget%28Stages%29.htm.)

Here are the four stages proposed by Piaget:

1. Sensorimotor, from birth to age 2. Children experience the world through movement and senses, using their five senses to explore the world.
2. Preoperational, from ages 2 to 7. Children respond to objects and events according to how they appear to be. They begin to use symbols.
3. Concrete operations, from ages 7 to 12 or so. Children begin to think logically, but mainly in quite concrete (non-abstract) situations and need practical (concrete) aids.
4. Formal operations, from age 12 onwards. Development of abstract reasoning, abstract thought, and logical thinking. People vary considerably in how far they progress in the formal operations stage.

Piaget was interested in math cognitive development as well as general cognitive development (Ojose, 2008). While Piaget's 4-stage theory was a major contribution to the field of cognitive development, it has gradually come under attack and has been modified to better represent more modern insights into cognitive development. Quoting from http://www.cliffsnotes.com/study_guide/Cognitive-Development-Age-711.topicArticleId-26831,articleId-26782.html:

> Piaget's model of cognitive development has come under increasing attacks in recent years. Modern developmentalists have frequently referred to experimental research that contradicts certain aspects of Piaget's theories. For example, cognitive theorists like Robert Siegler have explained the phenomenon of conservation as a slow, progressive change in the rules that children use to solve

> problems, rather than a sudden change in cognitive capacities and schemas. Other researchers have shown that younger and older children develop by progressing through a continuum of capacities rather than a series of discrete stages. In addition, these researchers believe that children understand far more than Piaget theorized. **With training, for instance, younger children may perform many of the same tasks as older children. Researchers have also found that children are not as egocentric, suggestible, magical, or concrete as Piaget held, and that their cognitive development is largely determined by biological and cultural influences.** [Bold added for emphasis.]

The idea of multiple intelligences discussed in Chapter 3 suggests the possibility that people might have different levels of cognitive development in different areas. Thus, a person might have a higher level of cognitive development in math as compared to in music, or vice versa. Opportunity and guidance are key aspects of cognitive development.

A recent Internet search of the expression *math OR mathematics "cognitive development"* produced about 435,000 hits.

Huitt and Hummel (2003) provides an introduction to Piaget's 4-stage theory of cognitive development and the role this theory provides in constructivist learning. In discussing the four stages, the article indicates:

> Data from adolescent populations indicates only 30 to 35% of high school seniors attain the cognitive development stage of formal operations (Kuhn, Langer, Kohlberg & Haan, 1977). For formal operations, it appears that maturation establishes the basis, but a special environment is required for most adolescents and adults to attain this stage.

The second sentence of the above quote is a nature versus nurture type of assertion. Researchers in cognitive development are faced by many of the same issues as researchers in IQ. Two of these are:

1. The (relative) roles that nature and nurture play in cognitive development.
2. Whether cognitive development is essentially domain-independent or is better described by a theory of "multiple" cognitive developments.

Math Maturity Food for Thought. Piaget's third stage, named Concrete Operations, described elementary school children. During the concrete operations stage, children begin to think logically. They come to understand that rearranging the items in a set does not change the number of items in the set. This stage is characterized by 7 types of conservation: number, length, liquid, mass, weight, area, and volume. See a short video at http://www.youtube.com/watch?v=j4lvQfhuNmg&feature=related.

During the time of this third stage, the school math curriculum moves relatively quickly into the use of abstract symbols and computations that are not solidly rooted in concrete situations. Think about the question of whether we are trying to push students too rapidly toward Formal Operations. Argue for and against the idea that this rapid push toward abstraction is the beginning of the familiar "I hate math." attitude.

IQ and Cognitive Development are closely related areas. An IQ test and a cognitive development test may well make use of some of the same questions or activities. IQ tests produce a number that tends to remain stable over time. This is because of the norming process discussed in Chapter 3. This norming process hides a child's steadily increasing intelligence gained through brain maturation and education.

A stage theory cognitive development test also produces a number (a stage level). However, this number tends to increase over time, as a person's brain matures and thus grows in intelligence and in cognitive development. This growth is not hidden by a norming process.

Michael Commons and Stage Theory

Michael Commons is a world leader in cognitive development stage theory. See http://en.wikipedia.org/wiki/Michael_Commons. Research by Commons and others of have expanded on and refined Piaget's work.

Commons and Richards (2002) provides a 15-stage Piagetian-type model of cognitive development. Their work has led to splitting a number of Piaget's scale points into multiple points and adding more stages at the high end. Quoting from the article:

> The acquisition of a new-stage behavior has been an important aspect of Piaget's theory of stage and stage change. Because of his controversial notions of stage and stage change, however, little research on these issues has taken place in the late twentieth century, at least among psychologists in the United States. The research that has taken place is being done by Neo-Piagetians. The Neo-Piagetians more precisely defined stage, taking each of Piaget's substages and showing that they were in fact stages. In addition, three postformal stages have been added.

The following is quoted from email from Michael Commons to David Moursund 5/10/09.

> The MHC [Model of Hierarchical Complexity] shows that stages are absolute and do not need in any way norms. Hierarchical Complexity is a major determinant of how difficult a task is. So stage and IQ should be quite correlated. My guess, is about a correlation of 0.5.
>
> …
>
> The evidence for stage change is a lot more clearly studied than IQ change. Most intervention[s] buy [that is, accept that strong interventions can produce] an increase of 1 or 2 stages at the most. I know of no studies showing more.

The first quoted part reemphasizes that IQ measures are normed and Cognitive Development measures on a Piagetian-type stage scale are not. Commons suggests that IQ and Stage level are moderately correlated.

The second quoted part provides Common's opinion about the extent to which concerted and extensive instruction and practice can raise a person on the 15-level scale. In terms of nature versus nurture, he seems to come down more strongly on the side of nature than on the side of nurture.

In brief summary, a person moves up in both intelligence and in cognitive development as his or her brain grows and matures, and through informal and formal educational experiences. The

levels that a person reach in intelligence and in cognitive development when growing up in a somewhat "average" or "typical" environment can be somewhat increased by more extensive, demanding, high quality informal and formal education. Tutoring, small classes, longer school days, and longer school years can all help students to move upward in both intelligence and cognitive development.

Stage Theory in Math

This section provides a five-level Geometry cognitive development scale based on the research of Dina and Pierre van Hiele. It also contains a six-level general math cognitive development scale that David Moursund created for use in his own work.

These scales start at a very low level. Over time, with increasing brain maturation and math-oriented informal and formal education, students move up these scales. Problem recognition, understanding, and solving lie at the heart of a stage theory. A general stage theory (such as Piaget's 4-stage theory) focuses on general, discipline-independent types of problems. A discipline-specific stage theory, such as those illustrated in this section, focus on problems within a specific discipline.

Geometry Cognitive Development

Piaget did a lot of research in developing his 4-stage model. Besides his general interests in cognitive development, he also had a particular interest in math cognitive development. This work suggests the importance both of a general stage theory and a stage theory and discipline-specific stage theories. Thus, one might have a stage theory for each of the multiple intelligences that Howard Gardner has identified.

Dina and Pierre van Hiele developed a Geometry cognitive development scale (see Figure 4.1) based on Piaget's work (van Hiele, 1959; Cathcart, et al., 2002). In their scale given Figure 4.1, levels 0 to 3 correspond to Piaget's levels 1 to 4.

Quoting from the Cathcart et al. (2000):

> In general, [in geometry] most elementary school students are at levels 0 or 1; some middle school students are at level 2. State standards are written to begin the transition from levels 0 and 1 to level 2 as early as 5th grade "Students identify, describe, draw and classify properties of, and relationships between, plane and solid geometric figures" (5th grade, standard 2 under Geometry and Measurement). This emphasis on relationships is magnified in the 6th and 7th grade standards.
>
> **Interestingly, the sixth National Assessment of Educational Progress report (1997) reported that "most of the students at all three grade levels (fourth, eight, and twelfth) appear to be performing at the 'holistic' level (level 0) of the van Hiele levels of geometric thought.** [Bold added for emphasis.]

If the last paragraph is an accurate assessment, it is a strong indictment of the level and quality of geometry instruction students were receiving in our schools at that time.

Level	Name	Description
0	Visualization	Students recognize figures as total entities (triangles, squares), but do not recognize properties of these figures (right angles in a square).
1	Analysis	Students analyze component parts of the figures (opposite angles of parallelograms are congruent), but interrelationships between figures and properties cannot be explained.
2	Informal Deduction	Students can establish interrelationships of properties within figures (in a quadrilateral, opposite sides being parallel necessitates opposite angles being congruent) and among figures (a square is a rectangle because it has all the properties of a rectangle). Informal proofs can be followed but students do not see how the logical order could be altered nor do they see how to construct a proof starting from different or unfamiliar premises.
3	Deduction	At this level the significance of deduction as a way of establishing geometric theory within an axiom system is understood. The interrelationship and role of undefined terms, axioms, definitions, theorems, and formal proof is seen. The possibility of developing a proof in more than one way is seen. (Roughly corresponds to Formal Operations on the Piagetian Scale.)
4	Rigor	Students at this level can compare different axiom systems (non-Euclidean geometry can be studied). Geometry is seen in the abstract with a high degree of rigor, even without concrete examples.

Figure 4.1. van Hiele five-stage Geometry cognitive developmental scale.

Notice that the van Hieles, being mathematicians, labeled their first stage Level 0. This is a common practice that mathematicians use when labeling the terms of a sequence. Piaget's cognitive development scale has four levels, numbers 1 to 4. The highest level in the van Hiele geometry cognitive development scale is one level above the highest level of the Piaget cognitive development scale.

Math Maturity Food for Thought. The Quoted material from Cathcart et al. (2002) suggests that at all grade levels, Geometry is being taught at a very low cognitive development level. Based on your insights into elementary school math, provide arguments for and against this assertion.

General Mathematics Cognitive Development

The scale in Figure 4.2 was created by David Moursund. It represents his current insights into a six-level, Piagetian-type, math cognitive development scale. The purpose of this scale is to facilitate discussion. It is not based on empirical research.

Stage & Name	Math Cognitive Developments
Level 1. Piagetian and	Infants use sensory and motor capabilities to explore and gain

Math sensorimotor. Birth to age 2.	increasing understanding of their environments. Research on very young infants suggests some innate ability to deal with small quantities such as 1, 2, and 3. As infants gain crawling or walking mobility, they can display innate spatial sense. For example, they can move to a target along a path requiring moving around obstacles, and can find their way back to a parent after having taken a turn into a room where they can no longer see the parent. As a child's brain matures, the child gets better at developing strategies for achieving aims. Even at Level 1, one can observe infants developing varying strategies related to dealing with various spatial and mobility problems.
Level 2. Piagetian and Math preoperational. Age 2 to 7.	During the preoperational stage, children begin to use symbols, such as speech. They respond to objects and events according to how they appear to be. The children are making rapid progress in receptive and generative oral language. They accommodate to the language environments (including math as a language) they spend a lot of time in, so can easily become bilingual or trilingual in such environments. During the preoperational stage, children learn some folk math (see http://iae-pedia.org/Folk_Math) and begin to develop an understanding of number line. They learn number words. They learn how to name the number of objects in a collection and how to count them, with the answer being the last number used in this counting process. A majority of children discover or learn "counting on" and counting on from the larger quantity as a way to speed up counting of two or more sets of objects. Children gain increasing proficiency (speed, correctness, and understanding) in such counting activities. Counting on is a strategy. One can learn strategies on their own and/or be taught strategies. In terms of nature and nurture in mathematical development, both are of considerable importance during the preoperational stage.
Level 3. Piagetian and Math concrete operations. Age 7 to 12.	During the concrete operations stage, children begin to think logically. In this stage, which is characterized by 7 types of conservation—number, length, liquid, mass, weight, area, and volume—intelligence is demonstrated through logical and systematic manipulation of symbols related to concrete objects. Operational thinking develops. This is the idea that some changes can be undone by reversing an earlier action. While concrete objects are an important aspect of learning during this stage, children also begin to learn from words, language, and pictures/video, learning about objects that are not concretely

	available to them. For the average child, the time span of concrete operations is approximately the time span of elementary school (grades 1–5 or 1–6). During this time, learning math becomes more constructivistic and is increasingly linked to having previously developed some knowledge of math vocabulary, symbols, and concepts. However, the level of abstraction in the written and oral math language quickly surpasses a student's previous math experience. That is, math learning tends to proceed in an environment in which the new content materials and ideas are not strongly rooted in verbal, concrete, mental images and understanding of somewhat similar ideas that have already been acquired. There is a substantial difference between developing general ideas and understanding of conservation of number, length, liquid, mass, weight, area, and volume, and learning the mathematics that corresponds to this. These tend to be relatively deep and abstract topics, although they can be taught in very concrete manners. Children in Level 3 are quite capable of developing math-related strategies on their own, and they are quite capable of learning strategies from teachers, peers, parents, and so on.
Level 4. Piagetian and Math formal operations. After age 12.	Starting at about age 12, or so, thought begins to be systematic and abstract. In this stage, intelligence is demonstrated through the logical use of symbols related to abstract concepts, problem solving, and gaining and using higher-order knowledge and skills. This level of math maturity supports the understanding of and proficiency in math at the level of a high school math curriculum and beginnings of understanding of math-type arguments and proof. Many students begin to take an Algebra 1 course before they are math developmentally ready to do so. That is, the level of abstraction and the level of math maturity assumed are quite a bit above their current levels. Piagetian and Math formal operations includes being able to recognize math aspects of problem situations in both math and non-math disciplines, convert these aspects into math problems (math modeling), and solve the resulting math problems if they are within the range of the math that one has studied. Such transfer of learning is a core aspect of Level 4.

	Level 4 cognitive development can continue well into college, and most students never fully achieve Level 4 math cognitive development. This is because of some combination of innate math ability and not taking cognitively demanding higher-level math courses or pursuing similar equivalent studies in other courses and/or on their own. Both nature and nurture are important in achieving Level 4. If a student's earlier math education is mainly "memorize and regurgitate with little understanding," then the student is likely to experience considerable difficulty moving upward through Level 4.
Level 5. Abstract mathematical operations. Moving far beyond math formal operations.	Mathematical content proficiency and maturity at the level of contemporary math texts used at the upper division undergraduate level in strong programs, or first year graduate level in less strong programs. Good ability to learn math through some combination of reading required texts and other math literature, listening to lectures, participating in class discussions, studying on your own, studying in groups, and so on. Solve relatively high level math problems posed by others (such as in the text books and course assignments). Pose and solve problems at the level of one's math reading skills and knowledge. Follow the logic and arguments in mathematical proofs. Fill in details of proofs when steps are left out in textbooks and classroom lectures.
Level 6. Mathematician.	A very high level of mathematical proficiency and maturity. This includes speed, accuracy, and understanding in searching and reading the research literature, writing research literature, and in oral communication (speak, listen) of research-level mathematics. Poses and solves original math problems at the level of contemporary math research frontiers. Thinks fluently in the language of mathematics.

Math Maturity Food for Thought. We know that children do not all develop physically at the same rate. Some are fast developers and some are slow developers. In sports, the faster physical development leads to greater success in sports. The greater success may well lead to greater personal effort and being on teams in which one gets good coaching. In some sense, success breeds success.

Now, think about the same idea, but in terms of cognitive development. Suppose that a student is somewhat slower than average in math cognitive development. The student is still expected to "compete" at the level of other students of his or her age. What do you think about the idea that "failure breeds failure?" Might our current educational system cultivate a culture of math education failure by getting many students in well over their heads—by moving them into relatively highly abstract math before they are cognitively ready for it?

Final Remarks

This chapter takes the point of view that a person's level of cognitive development may vary somewhat between disciplines. It emphasizes that in a particular discipline such as math, a substantial number of years of formal and informal education and extensive hard work are required to reach the higher levels.

These points of view are consistent with similar ideas in Math intelligence. People vary in their innate math gifts. When children with reasonably good innate math gifts are raised in a "rich and demanding" math education environment, they may well progress through the math curriculum about two or more times as fast as average students and begin to address significant math research problems at about the traditional age at which students complete high school.

Thus, one of the key ideas for improving math education is to immerse students in a demanding math curriculum that is designed to push the boundaries of their math cognitive development. Provide students with challenging problems; downplay rote memory and routine "busy body" work.

Activities and Possible Homework Assignments

1. **(For use with students.)** This activity makes use of a large number of colored cubes of a variety of colors. It is desirable to have five or more different colors, but the teacher can improvise if there are not this many different colors. In each case, students work in groups of two. If you have an odd number of students, one student helps you in the overall supervision and high-level observation process. In each pair, one student is the counter and serves as observer and reporter. The counter is given block-counting task, while the observers and reporters keep information about the processes used and possibly other data such as the accuracy of the count and the time to count the blocks. You, as the teacher, observe but do not intervene. You are looking for counting methodologies that may indicate a lower or higher level of math cognitive development and that may contribute to greater speed and accuracy. When a task or a round of tasks has been completed, you then lead a whole class discussion, but most of the talking is to be done by the observers and reporters. The discussion and debriefing might include noting different strategies used. It might ask if one strategy is apt to be more accurate or faster than another.

   Start by giving each pair of students a handful of blocks of mixed colors. For young students, give a small handful; for older students, give a larger handful. For young students, instructions can be given orally. For students who can read the instructions, present the instructions in writing.

   Version A. The counter is to count and record the number of blocks, striving for both accuracy and speed. The observer/reporter is asked to pay special attention to the counting methodology the methodology used. (For example, a student might count by 2's, selecting, moving away from the main pile, and counting all in a quick grab or swoop of the hand. After a round of counting, the two students in a pair switch roles. After two rounds of counting, discussion and debriefing occur.

Version B. The teacher names two colors. Each pair first moves all of their blocks into a nicely mixed up pile. The goal is for the counter to count and record how many blocks of each color are in the pair's pile of blocks. After completion, the teacher names two other colors, the students in a pair switch roles, and the counting task is repeated. In this situation, a student might begin by separating off into two distinct piles the two colors of blocks to be counted. A student might count one or both of the separate piles being crated during the separation process. [

Version C. Each pair of students moves their blocks into a nicely mixed up pile. The next goal is for the pair of students to work together to count and record how many blocks of each color that they have. A "smart" team may make use of the data obtained in Version B rather than recounting the four different colors of blocks counted in that activity. Indeed, if there are only five different colors of blocks, a pair can complete Version C merely by doing some arithmetic on data already gathered.

2. **(For use with students.)** Select two different versions of WordsWorth Plus described in Chapter 1. One should be chosen so that you believe it is suitable for use by the bottom third of your math class, and the other so that you feel it more suited to the top third of your class. Bottom third and top third should be based on your insights into the math intelligence, math cognitive development, and overall math maturity of the students you are teaching. As with (1) above, divide your students into pairs. Each student needs to have a piece of paper on which they write their name and on which they record their work.

   This activity may give you increased insight into the math cognitive development of your students. Some of yours students may find that the harder versions of the game are just not fun, while other find that the easier versions of the game are just not fun.

3. **(A possible homework assignment or discussion topic in a course.)** There are many different variations or levels of the WordsWorth Plus game discussed in Chapter 1. How can you tell if a particular variation or level is appropriate for a particular student or group of students? In thinking about and discussing this question, take into consideration ideas from math cognitive development theory.

4. **(A possible homework assignment or discussion topic in a course.)** Select a precollege grade level or precollege math course that you are quite familiar with. How can you tell if students at this grade level or in this course have the needed math intelligence and math maturity to succeed in the math instruction? What are some indicators you look for that suggest a student is "in over his or her head?"

5. **(A possible homework assignment or discussion topic in a course.)** Consider the profession of teaching and what an "experienced, mature" math teacher brings to his or her teaching that a first or second year teacher likely

does not. Try to identify some of the qualities and characteristics that might come with increasing math maturity and might be hastened though appropriate instruction and practice.

6. **(A possible homework assignment or discussion topic in a course.)** Chapters 2-4 of this book cover math maturity, math intelligence, and math cognitive development. Discuss:
    a. What are the similarities and the differences among these three ideas?
    b. Suppose a new student comes into your class midway through the first term. You are provided with the usual "transfer" records that are provided, and you get to talk to the student for just a very few minutes. What do you look for in the records and what do you talk about with the student to determine if the student has the background math knowledge, skills, and maturity to succeed in your class? What do you look for as the student joins your class and begins to participate in math-oriented class activities?

Chapter 5: Communication in the Language of Mathematics

> "The strongest memory is not as strong as the weakest ink." (Confucius; Chinese thinker and social philosopher, whose teachings and philosophy have deeply influenced Chinese, Korean, Japanese, Taiwanese and Vietnamese thought and life; 551 BC–479 BC.)
>
> "[The universe] cannot be read until we have learnt the language and become familiar with the characters in which it is written. It is written in mathematical language, and the letters are triangles, circles and other geometrical figures, without which means it is humanly impossible to comprehend a single word." (Galileo Galilei; Italian physicists, astronomer, mathematician, and philosopher; 1564–1642.)

This chapter explores math as a language. Certainly, math is not a general-purpose "natural" language such as English, Spanish, and Mandarin (Chinese). However, it is a language in which one can gain increasing levels of knowledge and skill in: speaking and listening; gesturing and interpreting gestures; reading and writing; creating and reading math-related diagrams, figures, and tables; and thinking. For a short video from the National Science Foundation, see *Learning the Language of Math* at http://nsf.gov/news/special_reports/math/language.jsp.

Communication is an act of conveying information. A sender and a receiver are involved. The communication might be between two people, between a person and a computer, between two computers, and so on.

Consider an attempted oral telephone communication between two people, one who knows only Mandarin Chinese and the other who knows only English. They will not be very successful in conveying information to each other.

Or, suppose that they both are fluent in oral English, but one is a world-class artist and the other is a world-class mathematician. The mathematician attempting to communicate math or the artist attempting to communicate art each face major challenges. The challenge is somewhat like a math teacher faces in attempting to teach math that is above the math knowledge and thinking level of a student. If the instruction is too much above a student's current level of knowledge and skills, the student will not be successful in building upon and integrating into his or her current level of knowledge and skills.

It is easy to see that communication between two people involves both having a common language and having some level of common understanding of the topic being communicated. An increasing level of math communication based on rote memory without understanding **is not** a sign of increasing math maturity. However, an increasing level of math communication skills based on an increasing level of math knowledge and understanding **is** a sign of increasing math maturity. Thus, for example, increasing capabilities to learn math by reading a math textbook, listening to a math "chalk and talk" lecture, of making use of interactive tutorials on the Web are all indications of an increasing level of math maturity.

Introduction

In math, we know that 3 + 5 = 8. We read this as three plus five **equals** eight or perhaps as three and five **equal** eight. However, the math symbol = does not have the same meaning as the English word equal in the statement, "all men are created **equal**" from the 1776 Declaration of Independence.

One way to think about this is that the math symbol = and the English natural language word equals are homonyms. That is, they are two different words with the same sound.

You know that in mathematics the Greek letter π is used to represent the length of the circumference of a circle whose diameter is 1. This is an important irrational number in math, with a value of approximately 3.1415927. An English language speaker, speaking in the language of mathematics, pronounces the Greek letter π (pi) the same as the word English word pie. (Also notice that "irrational number" has a quite precise meaning in the language of math.)

Note also that the math symbol π is much easier to write (and to say) than "length of the circumference of a circle whose diameter is 1." Similarly, think about *square root of 7*, $\sqrt{7}$, and $7^{1/2}$. Here, all three expressions have the same meaning. Notice, however, that the second and third expressions are much shorter than the first. The π and $\sqrt{\ }$ examples illustrate some of the types of shorthand in the language of mathematics.

When you watch and listen to a math lecture, you see and hear the lecturer write and speak in the language of mathematics. This special form of communication is, of course, embedded within the natural language being used by the speaker. Thus, the audience is faced by the task of understanding a presentation that is given using a combination of a natural language and the special language of mathematics. In essence, the speaker is bilingual and expects the audience to be bilingual. Moreover, the speaker expects the listeners to have sufficient understanding of math so that they can use the presentation to effectively review and then add to their knowledge and understanding of math.

Math Maturity Food for Thought. Think about your level of communication literacy in math, science, and music. In the previous paragraph we presented the idea that a person needs to be bilingual in math and a natural language to understand a math lecture. Similar statements can be made about understanding a music lecture or a science lecture. From this point of view, we are all multilingual. What are your thoughts on this idea? How might this idea affect how we teach children math, music, and science? For example, how might we teach for an increased level of math communication fluency, and what might be gained or lost if we did so?

The Discipline of Mathematics

As the totality of the accumulated knowledge of the human race has grown, scholars have found it desirable to divide this knowledge into somewhat loosely connected pieces. Thus, for example, in school students study Language Arts—which is further divided into speaking, listening, reading, and writing. Students study Mathematics—which is divided into sub disciplines such as arithmetic, geometry, algebra, statistics, and so on. Science has been divided into sub disciplines such as astronomy, biology, chemistry, physics and so on.

An academic discipline (area of study) or sub discipline can be defined by a combination of general things such as:

- The types of problems, tasks, and activities it addresses.
- Its accumulated accomplishments such as results, achievements, products, performances, scope, power, uses, impact on the societies of the world, and so on.
- Its methods and language of communication, teaching, learning, and assessment; its lower-order and higher-order knowledge and skills; its critical thinking and understanding; and what it does to preserve and sustain its work and pass it on to future generations.
- Its tools, methodologies, and types of evidence and arguments used in solving problems, accomplishing tasks, and recording and sharing accumulated results.
- The knowledge and skills that separate and distinguish among: a) a novice; b) a person who has a personally useful level of competence; c) a reasonably competent person, employable in the discipline; d) an expert; and e) a world-class expert.

Notice the emphasis on solving problems, accomplishing tasks, producing products, doing performances, accumulating knowledge and skills, and sharing knowledge and skills.

Math Maturity Food for Thought. Think about your knowledge, understanding, and skills in the discipline of mathematics from the point of view of the five bulleted items listed above. Then select a second discipline and do the same thing. Use these insights to do a compare and contrast between your strengths and weaknesses in the two disciplines.

Then repeat this activity, but this time thinking about precision of communication in the two different disciplines. Does it make sense to discuss the precision of communication in math versus the precision of communication in music?

Reading in Math

You have probably heard of the concept of *reading across the curriculum.* A recent Internet search of the quoted expression "reading across the curriculum" produced over 400,000 hits. Many preservice elementary school teachers receive instruction in this area.

The idea is simple enough. We want all students to learn to read. By the end of the third grade we expect many students to have learned to read well enough so that they can do fairly well in reading to learn. By about the end of the seventh grade, learning by reading is a major part of the design of the curriculum.

Well before students begin to learn to read, they have developed an extensive speaking and listening vocabulary. Thus, as they learn to read, much of the initial focus is on decoding the written symbols—translating them into the oral natural language they already know. The task is learning to decode, and then drawing on one's knowledge of the meaning of the (oral) representation of the written message in order to understand the written message.

It is much easier to learn to read in a highly phonetic language than in a language that is not very phonetic. It is still harder to learn to read in a non-alphabetic language such as Mandarin Chinese. While the language of math is embedded in natural languages, math can be thought of

a combination of a non-phonetic and a non-alphabetic language. It is not easy to gain a high level of fluency in communication in math.

Through instruction and practice students learn to read in their oral communication language. This oral communication language contains some math, science, and other discipline-specific vocabulary, knowledge, and understanding. Thus, the general-purpose introduction to reading includes making some progress in learning to read across the curriculum.

It is well understood that there is a difference between reading of rather general-purpose narratives and reading science, mathematics, and other disciplines. However, consider the following quoted material from **Chapter 4**, *Reading in Math* (Linda A. Hoover and James F. Nolan) from the book by Dupuis and Merchant (1993):

> According to Joan Curry, "Mathematics is a highly condensed system of language." Because reading in mathematics involves not only decoding words but also attaching literal meaning to mathematical symbols, and discerning the relationship between the two, teaching reading in the content area of mathematics is a particularly challenging task.
>
> Further complicating the integration of the two disciplines is the usual practice of teaching reading skills and processes separate from mathematics. Thus, the transfer from reading class to mathematics is difficult but essential.
>
> An especially difficult problem faced by students is that the reading level normally associated with a particular grade is often lower than the level necessary to comprehend that grade's mathematics text. Even the best readers can have difficulty making the transition from a narrative text, with its plot, characters, and setting, to a content-area expository text with a hierarchical pattern of main ideas and supporting details. Math texts are written in an especially terse and unimaginative style; they offer few verbal context clues to help with decoding meaning; and they lack the redundancy that makes writing easier to read. Another complicating factor is the variety of eye movements required to read math. The left-to-right rule often does not apply in reading number operations, as the set of diagrams on page 65 [of the document being quoted] makes graphically clear.
>
> The classic example of reading pitfall in math class is the "word problem." Students of mathematics who have not previously developed the necessary reading skills might be able to do the arithmetic if only they could read the problem with understanding.

The quoted material provides insight into why most precollege students do not learn to read math well enough to learn math by reading math. In essence, school math up through pre-algebra is taught using oral methods of instruction. Students develop some oral and written communication skills in math, but these skills tend to significantly lag the reading skills needed to learn new math topics by reading math.

High school math teachers are then faced by the dilemma of teaching math in the oral-tradition methods students are used to, versus emphasizing having students learn to learn math by reading. The result is that high school students vary widely in their math reading skills. Most do not develop math-reading skills appropriate to the demands inherent to the math they are studying. Their concept of reading a math book is to search for examples that look much like the

problems they are being asked to do as classwork and homework, and then to imitate the processes used in the examples.

Math Maturity Food for Thought. Think about the math reading skills of the math students you teach. Compare these reading skills with the math content you are teaching. Analyze the extent to which students are dependent on oral transmission methods to learn the math content you are teaching. Think about ways in which you might better help your students increase their math reading skills. Think about ways authors of math instructional materials can make their material more readable.

Native Speakers of a Language

We all understand the idea of a native language speaker of a natural language. Children who grow up in an oral natural language bilingual or trilingual environment readily develop oral fluency in two or three languages. Students learning an additional native language benefit by being taught by a native language speaker who can fluently listen, read, talk, write, and think in the language, and who is skilled in teaching the language.

This suggests a very important math education idea. In the same sense that we have native language speakers of a natural language who are skilled in teaching the language arts, we have some native math language speakers who are skilled in math. A child who grows up in such informal (home) and formal (school) math language environment has a significant advantage in gaining communication fluency in the math language.

From a math education point of view, on average both preschool environments and formal schooling at the elementary school level are presented by people who have a relatively low level of communication knowledge and skill in the language of mathematics. One way to partially overcome this weakness is to substantially increase the use of math education specialists in our elementary schools.

Another way is to significantly increase the amount of math-oriented communication in the everyday lives of children. There are many ways to do this. For example, when a parent is grocery shopping with a child, the parent can "think out loud" about the various costs and possible qualities of the goods, and the decision-making processes. An "at home" money education (allowances, spending, saving, planning) certainly involves a lot of communication in certain aspects of math.

Many games make use of math. The book you are currently reading provides a number of examples. Children learning these games, and then playing the games with adults and with other children, can gain considerable practice in various math-related types of communication.

Communication in a Multilingual World

You live in a world where there are a large number of different natural languages. In addition, many different disciplines such as math, music, and specific sports have developed their own languages. Thus, people have the opportunity to learn both natural languages and discipline-specific languages. For example, a person might be quadralingual, with a high level of communication skills in English, Spanish, math, and music.

Consider three different children, each growing up in typical middle class homes with parents of approximately the same level of education and income. In one of the homes, the parents are

strongly oriented toward music. They play musical instruments, spend a lot of time listening to and talking about music, and so on. In the second home, both parents are math teachers, one in a middle school and one in a high school. They like to talk about math, teaching math, learning math, solving math brain-teaser puzzles, and so on. In the third home, both parents are strongly oriented towards competitive sports. They engage in sports such as swimming, tennis, and golf. Considerable conversation in the home and considerable television watching is oriented towards such sports.

All three children will gain oral and gesture fluency in the natural language or languages of their parents. At the same time, each child will get a good start in a discipline-specific language. They will develop special insights into some of the culture, history, values, and content related to these discipline-specific languages.

When the three children interact with each other, they can communicate in their common natural language(s). They can also communicate—but likely at a lower level—in their discipline-specific languages. That is, because of the overlap of discipline-specific and natural languages, the children will all have some knowledge of and skills in communicating through each other's discipline-specific language(s).

As these three children attend school, they will gain a common core of educational experiences and they will gain in their total life experiences and general cognitive maturity. Thus, they will grow in their abilities to communicate with each other in their common natural language(s).

At the same time, the children may spend a lot of informal and formal education time in discipline-specific areas. For example, the child raised in a music-oriented home environment may be developing a steadily increasing level of oral and written music communication, knowledge, and performance skills and experience.

In school systems throughout the world, math is considered such an important discipline that students are required to study math year after year. All students are expected to learn to communicate in math, think in math, represent and solve problems involving math, creatively apply their math knowledge in non-math disciplines and problem areas, and so on. There are wide variations within our own country and throughout the world in how well education systems succeed in the educational endeavors.

The Language of Science

Logan (2000) considers science to be a language. The various sciences share many ideas in common, and all draw heavily on math. College majors in degree programs such as biology, chemistry, and physics all require a significant level of coursework in mathematics.

The sciences all include a focus on careful descriptions and predictions. Scientists pose and test theories. A key in this is the repeatability of the experiments. All sciences make use of careful measurements and on carefully reasoned arguments and methodologies in the processes of representing and solving the problems within their specific disciplines.

Table 5.1 lists the seven basic physical quantities used in the International System of Units (abbreviated SI). See http://en.wikipedia.org/wiki/Physical_quantity. Worldwide agreement on these definitions contributes to accurate communication among the scientists of the world.

Quantity	Unit name	Unit symbol
length	meter	m
mass	kilogram	kg
time	second	s
electric current	ampere	A
thermodynamic temperature	kelvin	K
amount of substance	mole	mol
luminous intensity	candela	cd

Table 5.1. International System of Units (SI).

Some Important Math-as-a Language Ideas

Here are some of the more important math-as-a-language ideas:

- It is easy to memorize math words and symbols without having a good understanding of their meaning. One sees this, for example, in young children as they are learning to say the counting words (numbers) and to recognize symbols 1, 2, 3, and so on before they can read one, two, three, and so on.
- Although one can spend a lifetime studying math and still learn only a modest part of the discipline, young children can gain a useful level of knowledge and skill via "oral tradition" even before they begin to learn to read and write. Moreover, preschool children learn some of the vocabulary and ideas of math communication as they learn their natural language or languages.
- Reading and writing are a major aid to accumulating information and sharing it with current and future people. This has proven to be especially important in math, because the results of successful math research in the past are still valid today.
- The language of mathematics is designed to facilitate very precise communication in and about math. This precise communication is helpful in examining one's own work on a problem, drawing upon the previous work of others, in collaborating with others in attempts to solve challenging problems, and in the accumulation of math knowledge of the human race.
- Our growing understanding of brain science is contributing significantly to our understanding of how one communicates with one's self in gaining increased expertise in solving challenging problems and accomplishing challenging tasks in math and in other disciplines. Metacognition and reflection about one’s work are very important aids to learning.
- Reading and writing help to overcome the limitations of one's short-term and long-term memory. Reading and writing using the language of math are powerful aids to the human brain as it works to solve complex, challenging math problems.
- Information and Communication Technology (ICT) has brought new dimensions to communication, and some of these are especially important in math. Printed books and other "hard copy" storage are static storage media. They store information, but they do not process information. ICT has both storage and processing capabilities. It allows the storage and

retrieval of information in an interactive medium that has some machine intelligence (artificial intelligence). Even an inexpensive handheld, solar-battery 6-function calculator illustrates this basic idea. There is a big difference between retrieving a book that explains how to solve certain types of equations, and making use of a calculator or computer that can solve all of these types of equations.

Final Remarks

Thinking about math as a language leads to increased insights into learning and teaching math. As an example, consider word problems in math. Many math teachers and students consider word problems to be the bane of their existences. Others revel in word problems, perhaps viewing them as "the next best thing to sliced bread."

As students are learning reading and writing, the initial focus is on words in their current oral vocabulary. Thus, as math and other disciplines introduce new words into a student's oral vocabulary, there needs to be emphasis on learning to read, write, and understand this discipline-specific vocabulary. This needs to continue apace with general instruction in reading and writing.

The next chapter in this book discusses the general idea of word problems in mathematics. In very simple terms, a word problem in math is a statement of a problem using natural language and/or discipline-specific languages, where the goal is to represent the problem in the language of mathematics, solve it, and interpret the results in terms of the language of the original problem statement.

Word problems are part of every academic discipline. An important aspect of math education is to help prepare students to deal with the math problem solving aspects of problems in other disciplines. To do justice to this goal, students taking math need to learn some general aspects of many different disciplines. Thus, for example, they need to learn and gain fluency in the language of science and in using the SI. Thus, students may well encounter SI both in math courses and in science courses.

Activities and Possible Homework Assignments

1. **(For use with students.)** As a whole class activity, have your students compile a list of math symbols and words that they have previously encountered. For each, work with the class to provide a relatively precise natural language definition. The idea is to create a math dictionary where the definitions are given in terms of the everyday language of your students. If your students do not include the words some of the math vocabulary you assume they have learned in previous math classes, suggest some of this vocabulary and use it to discuss how one easily forgets vocabulary that they do not routinely use.
2. **(For use with students.)** Have your students keep a math journal. Once a week (or, more often if this fits with your teaching preferences) have students write in the journal what they have learned in their informal and formal math studies and experiences since the last time they wrote in their journals. As a teacher, you can use the results to gain increased insight into the effectiveness of the informal and formal math instruction your students are receiving.
3. **(For use with students.)** Select a discipline other than math that your students are studying in classes you teach or that they are taking from other teachers.

Divide your class into small discussion groups, and have the groups talk about roles or uses of math in the discipline that you have named. Then do a whole class discussion and debrief, drawing on input from the various groups.

A variation on this activity is to have students do journal writing or written assignments on roles and uses of math in the various disciplines that they have studied. Another variation is to have students write about math in some of the games they play.

4. (**For use with students.)** As you are giving an oral presentation on a math topic to your class, stop once in a while and have a student explain in his or her words what you are saying. Call on a wide range of students as you repeatedly make use of this activity. This will provide you with some feedback on the effectiveness of oral presentation of math instruction.
5. **(A possible homework assignment or discussion topic in a course.)** Chapter 2 contains a list of various aspects of math maturity. Identify one or more items in that list that you feel are related to communication in the language of mathematics or in the intertwingling of natural language and math language? Discuss the relationships that you identify.
6. **(A possible homework assignment or discussion topic in a course.)** This chapter asserts that oral tradition is a widely used component or methodology of teaching math in our schools. Analyze your math-teaching methodologies. Pay particular attention to identifying ways in which you facilitate your students learning to read math, write math, and make use of math-related reference materials available in hard copy or on the Web.
7. **(A possible homework assignment or discussion topic in a course.)** Many young children learn to play a variety of card games and board games through hands-on, oral-tradition methods of instruction. Children learn the language, culture, and rules of the game, as well as how to play the game. You know that this is quite effective. Now, think about the "game" of mathematics. What are your insights into the advantages and limitations of teaching math in a hands-on, oral-tradition mode?

Chapter 6: Some Learning Theory

> "The mind is not a vessel to be filled but a fire to be kindled."
> (Plutarch; Roman historian; 46 AD–120 AD.)
>
> "If I have seen further it is by standing on the shoulders of giants."
> (Isaac Newton; English mathematician & physicist; Letter to Robert Hooke, February 5, 1675; 1642 – 1727.)

This chapter presents an introduction to a few teaching and learning theories. Such theories provide an approach to understanding what works and what does not work as we attempt to improve the effectiveness of teaching and learning processes.

For example, suppose that Student A has a goal of learning some content in order to pass an upcoming test, while Student B has the goal of learning some material because it will be routinely used in a profession he or she wants to pursue. Student B faces the task of transfer of learning to new situations he or she will encounter in the future. Effective and efficient teaching and learning for performance on an upcoming test is different than effective teaching and learning for long-term retention, use, and transfer of learning.

Here is one of the most important ideas in this book: An increasing level of math maturity is facilitated by teaching and learning for long-term retention, use, and transfer of learning.

Behaviorism

There are many different learning theories. Quite likely you have some familiarity with behaviorism. This theory has helped to shape our educational system.

John Broadus Watson (1878–1958) was an American psychologist who established the psychological school of behaviorism. (See http://en.wikipedia.org/wiki/John_B._Watson.)

However, most likely you associate behaviorism with the name B. F. Skinner (1904–1990). Burrhus Frederic Skinner was an American psychologist, author, inventor, and advocate for social reform. Quoting from http://en.wikipedia.org/wiki/B._F._Skinner:

> He came up with the operant conditioning chamber, innovated his own philosophy of science called Radical Behaviorism, and founded his own school of experimental research psychology—the experimental analysis of behavior. His analysis of human behavior culminated in his work *Verbal Behavior*, which has recently seen enormous increase in interest experimentally and in applied settings. He discovered and advanced the rate of response as a dependent variable in psychological research. He invented the cumulative recorder to measure rate of responding as part of his highly influential work on schedules of reinforcement. **In a recent survey, Skinner was listed as the most influential psychologist of the 20th century.** [Bold added for emphasis.]

Skinner stressed the need for positive reinforcement (rather than punishment) in learning. He argued that the main thing students learn from punishment is how to avoid punishment. Some of his work can be summarized by five principles of learning that he recommended.

1. **The learner needs immediate feedback.** Note that this feedback might come from external sources, such as a teacher, parent, peer, or teaching machine. The feedback might come from a great distance, via the Internet. It can also come from within the learner. Increasing maturity in any discipline is indicated by increasing ability to provide useful feedback to oneself.
2. **A task to be learned needs to be presented in small, manageable steps.** This general idea is the basis for much of the computer-assisted instruction materials that have come into wide use.
3. **The directions need to be very clear and repeated as many times as needed.** The learner needs to know where he or she is going in the learning task and in using the learning to solve problems and accomplish tasks.
4. **The instruction design/content should proceed from the quite simple** (where the learner achieves a high level of initial success) to the more complex. A somewhat different way of viewing this situation is to think in terms of the prerequisite knowledge and skills one needs to appropriately begin a new, challenging learning endeavor. See constructivism in the next section of this chapter.
5. **Provide ample positive reinforcement.** This relates to both intrinsic and extrinsic motivation and reinforcement.

Math Maturity Food for Thought. Think about a board game, card game, or other similar type of game—on or off of a computer—that you have learned to play reasonably well. Pay particular attention to the overall learning process and how you moved from being a complete novice to your current level of expertise. Analyze this learning experience in terms of the five principles of learning given above. Which of these principles seem most relevant to your learning experience, and which seem least relevant? Can you think of other principles you feel should be added to Skinner's list?

Constructivism

A healthy newborn child's brain contains some built-in knowledge and skills, and it has a great propensity to gain new knowledge and skills. Knowledge and skill is "stored" in collections of neurons, each neuron typically having one axon and multiple dendrites. Learning is stored through/by growth of new dendrites and strengthening of connections among dendrites from different neurons.

A healthy infant's brain contains far more neurons than it will have in adulthood. Many neurons will die off (be pruned) because they are not being used. Others will be repurposed, because the brain has a very high level of plasticity. Throughout life, there is a modest level of creation of new neurons.

In a somewhat simplistic point of view, new learning occurs by modifying or building on current learning. That is, learning causes changes to the neural structures that are in place. In that sense, new knowledge is built upon the existing knowledge structure.

Constructivism is a widely accepted learning theory that is consistent with the above brain analysis. Constructivism claims that knowledge is not passively received but actively constructed by the learner.

You are familiar with the idea that an instructional lesson containing new material has an assumption of certain prerequisite knowledge and skills. Often a lesson begins by a review of the prerequisites. This is a constructivist approach to designing and implementing a lesson.

David Ausubel fostered an instructional mechanism called *advance organizers*. Quoting Ausubel from http://tip.psychology.org/ausubel.html:

> These organizers are introduced in advance of learning itself, and are also presented at a higher level of abstraction, generality, and inclusiveness; …

One of the things that we standardly do in education is to help students build a considerable amount of knowledge and skill in certain areas, such as in reading and math. Reading and math are vertically-structured disciplines. Ideally, students in the fifth grade are building their reading and math knowledge and skills on a fourth grade (or higher) foundation. Unfortunately, in a typical fifth grade classroom there will be students whose reading and math knowledge and skills are far below the fourth grade level. For these students, there is a significant constructivism problem—the new instruction is too far above their current levels of knowledge and skills. In simple terms, the teaching is "over their heads."

A good teacher can accommodate a relatively wide range of prerequisite knowledge and skills. However, if the gap is too large, special instruction is needed. One-on-one tutoring or very small group instruction is effective in adjusting the level of instruction to better meet the prerequisite preparation of students.

For a quite different type of example, consider a team of basketball players with substantially different levels of physical and mental skill at shooting free throws. The players differ considerable in their overall nature-and-nurture-developed physical and mental free throw shooting capabilities. The coach is faced by the task of helping each player get better at shooting free throws.

The coach will provide individual instruction to players needing such individual instruction, some general instruction to the whole group, and a whole lot of opportunity to practice. A player will get some feedback from himself or herself, and perhaps additional feedback from the coach. Shooting free throws is such a complex physical task that no player will achieve absolute perfection, and so that all can benefit through continued practice and coaching.

To continue the basketball example, shooting free throws is only part of what it takes to be a good basketball player. Thus, the coach must make decisions on how much instruction and practice time should be spent on the various possible activities. How does one go about helping each player improve and at the same time constructing a good team and continuing to build its capabilities?

Math Maturity Food for Thought. Think about possible analogies between the basketball example and math education. What are similarities and differences? For example, suppose that a student is poor at arithmetic facts. How much of the teaching and learning effort for this student be devoted to arithmetic facts, and how much to learning the "whole game" of math?

An increasing level of math maturity is evidenced by increased proficiency in the "whole game" of math. Constructivist teaching and learning does not require absolute perfection of prerequisite knowledge and skills. The idea being recommended here is to spend more time engaging students in working on challenging problems and less time on near-perfect mastery of individual components. Think in terms of increasing expertise in the whole game of solving math problems rather than perfection in a small part of the prerequisites.

Situated Learning

Brown, Collins, and Duguid (1989) wrote a seminal article on situated learning, one of the important theories of learning. Quoting from the introduction to this paper:

> The breach between learning and use, which is captured by the folk categories "know what" and "know how," may well be a product of the structure and practices of our education system. Many methods of didactic education assume a separation between knowing and doing, treating knowledge as an integral, self-sufficient substance, theoretically independent of the situations in which it is learned and used. The primary concern of schools often seems to be the transfer of this substance, which comprises abstract, decontextualized formal concepts. The activity and context in which learning takes place are thus regarded as merely ancillary to learning pedagogically useful, of course, but fundamentally distinct and even neutral with respect to what is learned.
>
> **Recent investigations of learning, however, challenge this separating of what is learned from how it is learned and used. The activity in which knowledge is developed and deployed, it is now argued, is not separable from or ancillary to learning and cognition. Nor is it neutral. Rather, it is an integral part of what is learned. Situations might be said to co-produce knowledge through activity. Learning and cognition, it is now possible to argue, are fundamentally situated.** [Bold added for emphasis.]

Situated learning is a learning theory focusing on the situation or environment in which a particular learning activity occurs. For example, suppose that you are walking down a jungle path and you hear a particular sound that your brain/mind does not immediately recognize. You "freeze," carefully look around, and see a large snake.

Your brain/mind recalls that a friend of yours was seriously injured several weeks ago by a snake, and the description the friend gave seems to fit this snake. You immediately learn that the sound you have heard in this jungle trail environment is associated with a dangerous snake. Likely, this learning will last a lifetime. Moreover, the learning occurs very quickly—this is apt to be an example of one-trial learning.

Contrast this with being a middle-school student sitting in a classroom. You live in a large city, and there are few or no dangerous snakes within miles of your home. You are viewing a video discussing dangerous snakes. You see and hear a video of approximately the same scene as the jungle walker. However, the room you are in is hot and stuffy, you have just had lunch and you are sleepy, and the audio is turned up too high for your ears. What do you learn, and how long does this learning stay with you?

Math Maturity Food for Thought. Think about the traditional approaches to teaching and learning math. During the first few grades, much of the content students are learning has immediate usefulness to the students. For example, it is useful to be able to tell time and it is useful to be able to deal effectively and accurately with the types of numbers and number situations that come up in working with money.

However, in the upper grades of elementary school, content of the math curriculum begins to move beyond the immediate needs of many of the students. Think about this as the school learning situation and environment becoming increasingly separated from the student's outside of school needs and uses environment. Does this deviation from the recommendations from the theory of situated learning affect student learning in math?

Many math teachers attempt to address the situated learning issues described above. They do this by spending time trying to relate the school math curriculum topics to applications in other disciplines or applications outside of a school setting. Often, however, such transfer of learning is left to the student. The next section discusses transfer of learning.

Transfer of Learning

We want students to learn in a manner that allows them to use their learning in situations they will encounter in the future. These situations may be at work, at play, or in further schooling. We want students to be able to transfer their learning to these various situations they will encounter in the future. See http://iae-pedia.org/Transfer_of_Learning.

There are a variety of theories about transfer of learning. Stimulus-Response (S-R theory) provides a useful approach in some situations. Likely you are familiar with the work of Ivan Palov in using a S-R approach called classical conditioning in teaching dogs to have various behaviors. There is a great deal known about classical conditioning. See http://en.wikipedia.org/wiki/Classical_conditioning.

Near and Far Transfer

You may also be aware of the ideas of near transfer and far transfer. The "near and far" transfer of learning theory has been with us for a long time, and it is still widely discussed in the literature. Often the discussion is about training—for example, training a person to do a relatively specific job. Trainers want the learner to gain skill in a specific job task or job, but to also gain general knowledge and skills that transfer to other (not too similar) job tasks and jobs. In Industrial Age factories, many workers were trained to do a quite specific, repetitive task while their supervisors were trained and educated over a wide range of tasks and in the supervision of the trained employees.

The near-and-far theory of transfer suggested that some problems and tasks are so nearly alike that transfer of learning occurs easily and naturally. A particular problem or task is studied and practiced to a high level of automaticity. When a nearly similar problem or task is encountered, it is automatically solved with little or no conscious thought. This is called near transfer.

Learning to tie one's shoelaces using a bowknot provides a good example of near transfer. Once mastered to automaticity, there is easy transfer over time, to different pairs of shoes, and to different types of shoelaces. That is, the kinesthetic memory pattern of such shoelace tying is readily applied to new shoelace-tying situations.

In the near-and-far theory of transfer, learning and transfer tasks not falling into the near transfer category are called far transfer. There are many problems that are somewhat related, but that in some sense are relatively far removed from each other. For example, a student might learn to use the metric system in a math class, but not be able to transfer this knowledge and skill to a science class or to dealing with metric measurements encountered outside of a school setting.

The theory of near and far transfer does not help us much in our teaching. We know that near and far transfer occur. We know how to teach for near transfer. We know that some students readily accomplish far transfer tasks, while others do not. We know that far transfer does not readily occur for most students. The difficulty with this theory of near and far transfer is that it does not provide a foundation or a plan for helping a person to get better at far transfer and dealing with novel and complex problems. It does not tell us how to teach to increase far transfer.

Math Maturity Food for Thought. Think about near and far transfer in teaching and learning math. Many students approach studying for a math test to be a near transfer challenge. The goal is to memorize for quick recall and quick performance, focusing on the material that is likely to be on the test. Teachers realize that this is a poor approach to learning for understanding and for transfer of learning to novel problem-solving situations students will encounter in the future. Analyze your math teaching from the point of view of how it supports both near and far transfer. To what extent do you teach to the test and your students "study to and attempt to learn to" the test?

Low-Road/High-Road Theory of Transfer

The low-road/high-road theory of learning has proven quite useful in designing curriculum and instruction (Perkins and Solomon, 1992). In low-road transfer, one learns something to a high level of automaticity, somewhat in a stimulus/response manner. In essence, this is a description of a near transfer type of situation.

For an example, consider the situation of students learning the single digit addition or multiplication facts. This might be done via work sheets, flash cards, computer drill and practice, a game or competition, and so on. Most students require a lot of drill and practice over an extended period, along with subsequent frequent use of the memorized facts.

Moreover, many students find that they have difficulty transferring their arithmetic fact knowledge and skills from the (situated) school-learning environment to the (situated) outside of school environment. One of the difficulties is recognizing when to make use of the memorized number facts. In school, the computational tasks are clearly stated; outside of school, this is often not the case.

High-road transfer for improving problem solving is based on learning some general-purpose strategies and learning how to apply these strategies in a reflective manner. The strategy of breaking a complex problem into two or more less complex problems provides an excellent example. This strategy is often called the divide and conquer strategy

If you can solve the less complex problems you do so, and then put the results together to solve the original problem. If you have created a sub problem that you cannot solve, you may want to try breaking it into two or more still less complex problems.

Math Maturity Food for Thought. The divide and conquer strategy is a fundamental idea in having teams of people work together over a period of time to solve a complex problem or accomplish a challenging task. An individual or a team developing computer software routinely use this strategy. Think about the many thousands of people who worked on various parts of the task of sending people to the moon and safely returning them to earth. Now, think about examples from your own life in which you employ the divide and conquer strategy. Think about how you prepare your students to work in teams that solve problems that are too large and complex for one individual to solve.

The *build on your previous work strategy* is another excellent candidate to have in your repertoire of high-road transferable problem-solving strategies. Think of situations in which you encounter a new problem and solve it by drawing on your previously-gained knowledge and skills. You can get better at this through conscious practice and reflection as you solve problems.

And, of course, your brain naturally uses the "build on your previous work" strategy. When your conscious brain encounters a problem and passes it on to your subconscious, your subconscious uses a pattern matching approach. It tries to match the pattern or patterns in the problem to patterns developed through previous learning.

A closely related strategy is to build on the previous work of others. You do this when you make use of a reference book or a textbook. You do this when you use a calculator or a computer.

Here is a strategy for teaching and learning high-road transfer of learning of problem-solving strategies:

1. Identify the generalizable strategy that is being illustrated and used in a particular problem-solving situation.
2. Give the strategy a name that is both descriptive and easily remembered.
3. Working with your students, identify a number of different examples in other disciplines and situations in which this named strategy is applicable.
4. Have students practice using the strategy in a variety of areas in which it is useful, and where students have appropriate general and domain-specific knowledge.
5. In your everyday teaching, you will frequently encounter situations in which a particular problem-solving strategy is applicable and you have previously helped your students gain some initial expertise in using the strategy. Take advantage of such situations by clearly naming the strategy (or, asking your students to name the strategy) and working with your students to refresh their memories on use of the strategy in a variety of situations.

Final Remarks

Math education in schools tries to achieve an appropriate balance between the rote memory and automaticity of low-road transfer, and the critical thinking and understanding used in high-road transfer. One approach is by making extensive use of word problems or story problems.

In word problems, the specific problem-solving tasks to be performed are not made explicit. In essence, the problem solver is faced by the challenge of breaking the original problem into a collection of sub problems, solving the sub problems, and putting the results together to solve the original problem. From that point of view, the divide and conquer strategy is a routine approach to solving word problems.

Of course, divide and conquer is but one aspect of problem solving. There are many problem-solving strategies that are useful both in math and in other disciplines. High-road transfer of learning is an important approach to teaching and learning in every academic discipline.

The next chapter of this book focuses on roles of games in math education. A game has materials and rules that define a situation in which players solve problems and accomplish tasks.

Activities and Possible Homework Assignments

1. **(For use with students.)** Select a topic from the math lesson or unit you are currently teaching and that you have just covered in class. Have your whole class work together to give examples of uses or applications of this math topic to situations outside of the math class. Use this activity to stress the idea of transfer of learning. This is an activity that can be built into every math lesson or unit that you teach. Use it to explicitly teach your students some widely used problem-solving strategies and the idea of high-road transfer of learning.
2. **(For use with students.)** Select a topic from the math lesson or unit you taught a month or more ago. As a whole class activity or as an individual student activity, have students give examples of uses of this math content that they have made outside of the math class since the topic was covered in class. This activity ties in with the idea of long term retention and the need for periodic use of a topic if it is going to remain fresh and easily recalled from one's long term memory.
3. **(For use with students.)** Carry on a whole class discussion with your students about their insights into how to learn something. (You might first have students write a little about how they go about learning something that they want to learn.) In essence, each student has some personal theories of learning. The purpose of this discussion is for you and your students to gain increased insights into differing but widely used theories of learning.
4. **(A possible homework assignment or discussion topic in a course.)** Explain constructivism in your own words, using examples from your own life as a learner and from your teaching experiences. Make sure you include the idea of "prerequisite knowledge and skill" learning. Think of your intended audience as parents of the students you teach, and your objective as trying to help them understand the relevance and importance of the constructivism theory of learning.
5. **(A possible homework assignment or discussion topic in a course.)** Analyze the content, pedagogy, and assessment from a situated learning point of view for a recent unit of instruction that you have taught. In this analysis,

you might want to try to see the unit of instruction through the eyes of your students and their current "life situations."

6. **(A possible homework assignment or discussion topic in a course.)** What do you think your students should know about learning theory? Select a learning theory that you feel is important in teaching and learning. Explain and illustrate it in a manner that will communicate effectively to the students that you teach. Give arguments for and against having your students learn about some of the theories of learning.

Chapter 7: Math Word Problems

> To understand mathematics means to be able to do mathematics. And what does it mean doing mathematics? In the first place it means to be able to solve mathematical problems. (George Polya; Hungarian mathematician; 1887–1985.)
>
> "In the book of life, the answers aren't in the back." (Charles M. Schulz; American cartoonist speaking through the voice of his comic strip character Charlie Brown; 1922–2000.)

Math is a required part of the precollege school curriculum because it is a powerful aid to representing and helping to solve a wide variety of problems that people encounter in their every day lives. Math is also required because it is an important tool and language in helping to represent and solve problems in many different non-math disciplines that students are studying or may want to study in the future.

Notice the emphasis on problem solving in the quotation from George Polya. This chapter focuses on math problem solving, with special emphasis on solving math word problems.

Examples

Typically, students in a math course are asked to solve two different kinds of problems.

1. A problem that is clearly stated in the language of math. Examples include:
   A. Find $1 + 2 + 3 + 4 + 5$.
   B. Solve for x in the equation $5x + 6 = 41$.

   In these two examples, the word "find" and the phrase "Solve for x" can be considered as parts of the language of mathematics.
2. A problem that is stated in a natural language, perhaps also using some words from the language of mathematics. For examples include:
   A. Mary is a year older than her sister. The sum of their ages is thirteen. How old is Mary's sister?
   B. As a fund-raiser, the students, teachers, and Parent Teachers Organization have all agreed to collect pennies. Their goal is to collect enough pennies so their total weight is the same as the 75kg school principal. If they successfully complete this task, roughly how much money will they have collected?

The second type of problem is called a word problem or a story problem. One way to solve a math word problem is by representing it clearly in the language of mathematics, solving the resulting math problem, and translating the results back into the context of the original (word) problem. The two sisters problem can be solved that way, using Algebra. Note however, it can also be solved by a little use of trial and error (guess and check) by a student who has no

knowledge of algebra. Guess and check is a strategy useful in solving many math and non-math problems.

Food for Mathematical Thought. Notice that the "pennies" problem will require students to determine the weight of a penny. This is needed information that is not included in the statement of the problem. Moreover, since pennies vary in weight due to varying year of mintage and wear, there is an added challenge in this problem. The goal is to make a good estimate for an answer.

Notice also that we used the word *weight* in the pennies problem. A more precise statement of the problem would have used the word *mass*. Can you explain why?

Finally, many countries have a coin that is called a penny. Do you think that they all have the same mass?

The above discussion illustrates the challenge of precisely stating a word problem. Many of the word problems stated in math books are poorly stated. For example, what does it mean to say that Mary is a year older than her sister? Does this she is **exactly** one year older (to the nearest day, the nearest hour, the nearest second) than her sister? The phrase "a year older" is usually taken to be an approximation. If that is the intent in the problem stated earlier, then there are many possible answers.

What do you think the students you teach should be learning about the precise statement of word problems? For example, should they learn to recognize when a problem is poorly stated? Progress in precise math-related communication is an indicator of increasing math maturity.

Some General Issues

Each discipline focuses on particular types of problems and tasks. Thus, each discipline has its own ideas as to what constitutes a problem, task, or activity that helps define the discipline and that is important for students to learn about when studying the discipline.

Consider various disciplines, such as art, business, history, music, and physics. Each makes use of math. This presents a major challenge to a general education system. Should we have courses in math in art, math in business, math in music, and so on? Or, should teaching math be the province of math teachers, with the expectation that students will transfer the math learned in math classes into the various other disciplines that students study?

The second approach would seem to be more efficient. Over the years, math instruction at the K-12 levels has evolved so that the content being taught more or less satisfies the math-oriented needs of students taking non-math courses. The "more or less" statement is because of:

1. Transfer of learning from one discipline to another is difficult. Many students find it quite difficult to transfer their math learning to problem situations in disciplines outside of math.

   It is possible to teach in a manner that increases transfer of learning. Such teaching and learning is especially important in disciplines such as math that are useful in so many other disciplines.

2. Many students forget important math topics they have studied in math courses, so that if the math is needed in a subsequence non-math course, the teacher feels obligated to teach it.

   When studying in a particular discipline, how rapidly one forgets and the amount that one forgets varies widely with the student, the method of instruction, the method of learning, and so on. Math teachers can address some of these issues by teaching for understanding rather than for passing tests by rote memory, and by giving students instruction in and experience in relearning math they have forgotten. A key component of this is helping students learn to read math well enough so that their level of math reading knowledge and skill is useful in relearning math they have forgotten and in retrieving information about math topics they have forgotten, and in learning to make use of resources such as the Web.

   Still another possibly useful approach is to make available detailed descriptions of math courses easily available to the non-math community. These descriptions need to be written in a manner that facilitates teachers in non-math disciplines quickly being able to find out what students are "supposed to know" from their previous math coursework.

3. It often happens that a non-math course at the precollege level encounters math topics that students have not studied in their math courses.

   Computers have exacerbated this difficulty. As an example, a relatively young student can learn to make use of computer graphics software in dealing with editing photos and video, designing and developing graphic images, and doing animation. The math involved "behind the scenes" in all of these activities is typically well above what the students have studied—and, indeed, well above what the teacher has studied. However, with proper instruction and examples, the students can develop a useful intuitive understanding of the underlying math and relate it to math they have studied.

   Moreover, such situations may well be such that it is quite appropriate for a teacher in a non-math discipline to review and/or teach some of the underlying math. The math will likely be presented using the vocabulary and problems of the non-math discipline.

This is a book oriented toward teachers of math. The first two of the numbered suggestions given above are things that all math teacher can be doing. This chapter will help.

Definition of Problem

There are a huge number of publications on the general topic of problem solving. A recent Internet search of the quoted expression "*problem solving"* produced more than 22 million hits.

Each discipline tends to use its own vocabulary and types of examples in saying what constitutes a problem and in discussing problem solving. In this book, we use the term **problem solving** to include all of the following activities:

- **Question situations:** recognizing, posing, clarifying, and answering questions.
- **Problem situations:** recognizing, posing, clarifying, and solving problems.

- **Task situations:** recognizing, posing, clarifying, and accomplishing tasks.
- **Decision situations:** recognizing, posing, clarifying, and making good decisions.
- **Thinking:** using higher-order critical, creative, wise, and foresightful thinking to do all of the above. Often the results are shared, demonstrated, or used in a product, performance, presentation, or publication.

This broad definition is intended to encompass the critical, creative, and higher-order thinking activities in every discipline. An artist, historian, mathematician, musician, poet, and scientist all do problem solving.

The essence of a problem and problem solving is illustrated in Figure 7.1.

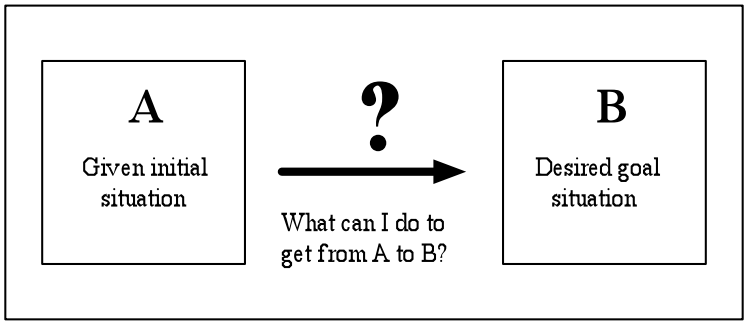


Figure 7.1. Problem-solving—how to achieve the final goal?

In the academic literature, there is a reasonable amount of agreement on what constitutes a problem. You (personally) have a problem if the following four conditions are satisfied:

1. You have a clearly defined given initial situation.
2. You have a clearly defined goal (a desired end situation). Some writers talk about having multiple goals in a problem. However, such a multiple goal situation can be broken down into a number of single goal problems.
3. You have a clearly defined set of resources that may be applicable in helping you move from the given initial situation to the desired goal situation. There may be specified limitations on resources, such as rules, regulations, and guidelines for what you are allowed to do in attempting to solve a particular problem.
4. You have some ownership—you are committed to using some of your own resources, such as your knowledge, skills, time, money, and so on to achieve the desired final goal.

The four-part definition of a problem does not tell you how to solve a problem. But, it does contain some hints. For example, to solve a problem it is necessary to know what problem you are trying to solve (component 1 of the definition) and to know what constitutes a solution (component 2 of the definition).

Component 3 concerns the resources you have available. Do you have a high level of expertise and experience in the problem area? Do you have access to problem-solving aids such

as needed tools—including a computer and Internet connectivity? Do you have friends who will help you, or money to hire assistance and to purchase other resources?

Component 4 is omitted in some people's definition of a problem. That is a serious mistake. If you are interested in (invested in, committed to, have intrinsic motivation) solving a particular problem, you will devote your time, energy, brainpower, and other personal resources to the task. If you "couldn't care less" (which often happens in school classwork and homework problems) then chances of success in solving the problem are greatly decreased.

These four components of a well-defined (clearly-defined) problem are summarized by the four words: givens, goal, resources, and ownership. If one or more of these components are missing, you have an ill-defined problem . This is called a *problem situation.* It is not a well-defined problem. An important aspect of problem solving is realizing when you are dealing with an ill-defined problem situation and working to transform it into a well-defined problem.

Math Maturity Food for Thought. There are many very challenging problem situations, such as world hunger and disease, war, homelessness, and sustainability. You may be interested in and concerned about these various situations. For you, each is a *problem situation* until you have very a clearly stated initial situation, goal, resources, and ownership. Select some problem situation that concerns you and develop it into a carefully defined problem in which you have considerable ownership. Think about what you do to help your students gain skill in working with ill-defined problem situations and developing them into carefully defined problems.

Polya's Six-Step Problem-solving Strategy

George Polya was one of the leading mathematicians of the 20th century, and he wrote extensively about problem solving. One of his major contributions came from a careful analysis of good ways to approach and attempt to solve a challenging math problem. The six-step strategy provides a general way to approach to attempting to solve a challenging problem. It is not a step-by-step procedure that is guaranteed to solve a problem.

1. Understand the problem. Among other things, this includes working toward having a well-defined (clearly defined) problem. You need an initial understanding of the Givens, Resources, and Goal. This requires knowledge of the domain(s) of the problem, which could well be interdisciplinary. You need to make a personal commitment to solving the problem.
2. Determine a plan of action. This is a thinking activity. What strategies will you apply? What resources will you use, how will you use them, in what order will you use them? Are the resources adequate to the task?
3. Think carefully about possible consequences of carrying out your plan of action. Place major emphasis on trying to anticipate undesirable outcomes. What new problems will be created? You may decide to stop working on the problem or return to step 1 as a consequence of this thinking.
4. Carry out your plan of action. Do so in a thoughtful manner. This thinking may lead you to the conclusion that you need to return to one of the earlier steps. Note that this reflective thinking leads to increased expertise.

5. Check to see if the desired goal has been achieved by carrying out your plan of action. Then do one of the following:
    A. If the problem has been solved, go to step 6.
    B. If the problem has not been solved and you are willing to devote more time and energy to it, make use of the knowledge and experience you have gained as you return to step 1 or step 2.
    C. Make a decision to stop working on the problem. This might be a temporary or a permanent decision. Keep in mind that the problem you are working on may not be solvable, or it may be beyond your current capabilities and resources.
6. Do a careful analysis of the steps you have carried out and the results you have achieved to see if you have created new, additional problems that need to be addressed. Reflect on what you have learned by solving the problem. Think about how your increased knowledge and skills can be used in other problem-solving situations. (Work to increase your reflective intelligence.)

Many people have found that this six-step strategy for problem solving is worth memorizing. As a teacher, you might decide that one of your goals in teaching problem solving is to have all your students memorize this strategy and practice it so that it becomes second nature. Help your students to make this strategy part of their repertoire of high-road strategies. Students will need to practice it in many different domains in order to help increase transfer of learning. This will help to increase your students' expertise in solving problems.

Many of the steps in this six-step strategy require careful thinking. However, there are a steadily growing number of situations in which much of the work of step 4 can be carried out by a computer. Remember, a computer is a resource that is a valuable aid to problem solving. The person who is skilled at using a computer for this purpose may gain a significant advantage in problem solving, as compared to a person who lacks computer knowledge and skill.

Easy, Hard, and Unsolvable Problems

Perhaps you and a friend are faced by the same problem. The problem might be very easy for you to solve and very difficult for your friend to solve, or vice versa. Through education and experience, a problem that was difficult for you to solve may become quite easy for you to solve. Indeed, it may become so easy and routine that you no longer consider it to be a problem.

This is a key idea. Through study, practice, and access to appropriate tools, you can increase your level of expertise in solving a particular category or type of problem. One way to think about informal and formal education is that the goal is to increase your level of expertise in various problem-solving areas by converting problems that you once considered challenging into mere routine exercises.

Another key idea is that there is nothing in the definition of a problem that suggests how difficult or challenging a particular problem might be for you. One problem might be quite easy to solve while a somewhat similar problem will have no solution (be impossible to solve).

Solvable and Unsolvable Problems

Here is a problem that interested and engaged many scientists in the past:

Find some chemicals or compounds, none containing atoms of gold, that when mixed together and heated over a stove or gas burner will produce gold.

A great many alchemists have tried to solve this problem. They had ownership—success would make them rich and famous. However, this problem is not solvable subject to the constraint paced on the heating apparatus. A supernova explosion of a star can produce gold. (See http://www.sjaa.net/eph/0706/f.html.)

Math Maturity Food for Thought. How do you react to a sentence such as the assertion that a particular problem is not solvable? Did you just blip over the gold production assertion, or did you challenge the assertion? In the latter case, did you think about the "big bang" (a theory about the creation of our universe) that initially produced the lighter elements? It was only much later that heavier elements such as gold were created in the interiors of stars. One indication of an increasing level of maturity in math or any other academic discipline is an increasing ability to challenge assertions and to seek out evidence (stored in your own brain or elsewhere) that perhaps supports or disproves the assertion.

Think about this in terms of the teaching of math. Math can be taught as a set of assertions that students are to accept, memorize with little understanding, and then use to solve problems. This style of teaching does not facilitate learning to question and doubt assertions. It is also one of the reasons why many students quickly forget much of the math they have studied.

Here is a simple math problem that has no solution. Find two positive even integers whose sum is an odd integer. Just because one has a clearly defined problem does not mean that one has a solvable problem. No matter how strongly you want to find two positive even integers whose sum is an odd integer, you will not be able to solve the problem.

Indeed, a good way to handle such a problem is to develop a convincing argument or proof that it has no solution. It may seem obvious to you that there is no solution. Remember, however, that there are an infinite number of integers. How does one go about proving that there is no solution when there are an infinite number of integers? Just giving a few examples and then claiming that the result is "obvious" does not constitute a proof.

Here is a very similar problem. Find two positive odd integers whose sum is an even integer. This problem has an infinite number of solutions.

Can you think of (create) a problem that has exactly five solutions? Here is an example from math. Find a positive integer that is greater than 2 and less than 8.

In brief summary, it is easy to find clearly defined problems that have no solution. It is easy to find clearly defined problems that have exactly one solution, and it is easy to find others that have two solutions, three solutions, or still more solutions. Thus, one of the things to think about when faced by a clearly defined problem is that it might not have a solution, it might have exactly one solution, or it might have more than one solution.

These same statements hold for problems in other disciplines. Many people seem to think that if one could use methods from math and the sciences, any problem can be solved. That is just not the case.

Math Maturity Food for Thought. The chances are good that when you were learning math at the precollege level, you "learned" that the goal in solving a problem is to get "the" answer. Reflect on this incorrect teaching and whether you are also guilty of it. Did your teachers help you to understand that a problem might have no solution, exactly one solution, or more than one solution?

Give some examples of problems in non-math disciplines that have many different solutions. For example, consider architecture (designing an energy efficient building), gardening (designing a lovely garden), and so on. How might you use such examples and transfer of learning to help students break the habit of talking about "the " solution to a problem?

Quality or Beauty of a Solution or Solution Process

Suppose that you are at an art exhibit and you see a number of different paintings of a sunset. Each painting is a solution to the task: "Paint a sunset." But, some are more creative, elegant, or beautiful than others.

Somewhat similarly, suppose that students are asked to write an essay comparing and contrasting two forms of government. A student's essay is a solution to the assigned task. But, some essays will be much better than others.

Mathematicians talk about the beauty of math. They talk about a proof or solution process being beautiful, elegant, creative, easy to understand, and so on. See, for example, http://users.forthnet.gr/ath/kimon/.

Thus, mathematicians are interested both in results (a correctly solved problem; a correctly proved theory) and the various representations or processes displayed in solving a problem or proving a theorem. That is one reason why math teachers stress, "show your work." One indicator of an increasing level of math maturity is an increasing ability to effectively communicate one's work and thus display the elegance and beauty of thought involved in the work.

In school, students are presented with problems that have been previously solved by millions of other students. In many cases, it is possible to use information retrieval techniques to look up a solution and a solution process. Indeed, it is increasingly possible to merely tell the problem to a computer system (for example, Wolfram Alpha at http://www.wolframalpha.com/) and the computer system will both solve the problem and explain the process it used to solve the problem.

As teachers, we must help students to understand that the goal **is not to just** "get an answer." Much more important goals include learning to learn math, increasing one's level of math understanding and maturity, fostering math creativity, and increasing one's insight into the beauty and elegance of mathematics.

Math Maturity Food for Thought. You are familiar with beauty in many different disciplines. As a math teacher, what do you do to help your students learn about beauty, elegance, and creativity in math? Think in terms of transfer of leaning to and from the discipline of mathematics.

Dealing With Problem Situations

Many things that people label as problems are actually ill-defined (poorly-defined) problem situations.

Look back at Polya's six-step strategy. The first step is to work toward having an understandable, clearly-defined problem. Start by analyzing the problem situation in terms of the four components of a clearly defined problem.

Here is an example. You turn on a television set and you view a brief news item about the homeless people in a large city and the starving children in a foreign nation. The announcer continues with a news item about students in our schools scoring poorly on an international test, relative to those from some other countries. The announcer presents each news item as a major problem. However, it is much more accurate to say that these are problem situations.

To determine if they are clearly defined problems from your point of view, you can ask yourself four questions:

1. Is there a clearly defined given initial situation? Do I really know the facts? Can I check out the facts through alternative sources that I feel are reliable?
2. Is there a clearly defined goal? Is it really clear to me how I would like things to be? Are there a number of possible goals? Which goal or goals seem most feasible and viable? Will I be able to tell if a goal I select has been achieved?
3 Do I know what resources are available to me that I could use to help achieve the goal? In addition, are there rules, regulations, and guidelines that I need to know about as I work to solve this problem?
4. Do I have ownership—do I care enough to devote some of my own resources? Am I willing to spend some of my own time, money, and mental and physical energy on achieving the goal?

If you can answer, "yes" to each of these questions, then you (personally) have a formal, clearly defined problem.

Often, your answer to one or more of the questions will be "no." Then, the last question is crucial. If you have ownership—if you really care about the problem situation—you may begin to think about and clarify the problem situation. You may decide on what you feel are appropriate statements of the givens and the goal. You may seek resources from others and make a commitment of your own resources. You may then proceed to attempt to solve the problem.

Finally, you need to know that just because you have a clearly defined problem does not mean that you (or indeed, anyone else) can solve it. As noted earlier, many problems have no solution.

In math courses, the word problems that students are asked to solve are almost always problems that have clearly defined initial situations and goals. While resources are not typically made explicit, they include using one's knowledge and skill, and a restriction "do your own work."

You can help your students to gain an increased level of math maturity by presenting them with challenging math problem situations. Include in this mix of problem situations some that

easily lead to clearly defined problems, but in which the clearly defined problems may have no solution, or multiple solutions where it is difficult to tell if one solution is better than another.

Math Maturity Food for Thought. Have you ever given your students word problems that have no solution? How often would you need to do this in order for your students to learn that there are problems that are clearly defined but that have no solution? Do you think this would be a good thing to teach in a math course? Think about some ways to teach this idea in a manner that transfers from math problem solving to problem solving in other disciplines.

Good Math Word Problems

This section contains a list of some characteristics that a good math word problem might have. One way to measure the quality of a math word problem is to see how many of these characteristics it has. You should not expect a math word problem to satisfy all of the criteria listed below. However, the recommendation is that students should regularly be asked to work with word problems that, cumulatively, give them substantial practice that incorporates all of these recommendations.

1. The problem is challenging to a relatively wide range of students and can be approached and solved in a variety of ways. Likely, such a problem will be quite a bit more challenging to some students than to others, but a wide range of students can make progress on and/or solve the problem thorough perseverance.

   We live in a society in which many children and adults seek (indeed, often demand) instant gratification. Contrast the idea of instant gratification with that of delayed gratification that comes after significant effort over a period of time. One indicator of an increasing level of math maturity is the increasing ability to work diligently on a math problem over a sustained period of time.

2. The problem makes use of content knowledge from both math and from disciplines outside of math. That is, the problem is interdisciplinary and serves as a vehicle for the student to gain increased knowledge and skill in solving math-related problems across the curriculum and in learning more about the math-related problems from these disciplines.

   A good problem may well require a student to do "library research"—that is, do some reading and information retrieval in the discipline or disciplines that the problem is embedded in.

   Our school system has found it desirable and convenient to divide the curriculum into discipline-specific pieces. However, the real world is not that way. Problems in the real world tend to be multidisciplinary.

   Many students have a great deal of difficulty dealing with multidisciplinary problems. They have had little experience or instruction in transferring their math knowledge and skills to other discipline areas. Thus, they have a low level of math maturity in this very important aspect of moving toward a higher level of math maturity.

3. The problem gives students practice in reading with understanding both in math and across the curriculum. The content and the needed underlying understanding in such a problem provide one approach to checking a proposed solution. Does the proposed solution make sense?

   The "sense making" question is critical to problem solving in every discipline. As a simple example, suppose a student is presented with a problem that concerns the ages of various people. Perhaps the student uses algebra, develops some equations, solves the equations, and produces an answer that one of the people is minus three years old. Does that make sense? Or, perhaps the student arrives at a solution in which one of the people is 400 years old. Does that make sense?

   One of the indicators of increasing math maturity is a student's ability to use sense making and understanding as an aid to detecting errors in thinking and problem solving.

4. The problem embodies clear instructional and learning goals that are consistent with and supportive of the overall math goals of the course and the school's math curriculum. One way to approach this question is to see how the problem fits in with the national goals.

   Math education and math problem solving need to be viewed as a coherent whole rather than as isolated disconnected pieces. One indicator of increasing math maturity is learning to view and understand the world from a mathematical point of view—that is, to learn to think about and analyze problems drawing on the math that one has learned.

5. The problem requires a student to draw on his or her cumulative math knowledge and skills. Contrast this with the types of classwork and homework that mainly require students to make use of the math content covered in the lesson of the day or the material covered in the past few days.

   One indicator of increasing math maturity is increasing ability to solve problems that require drawing on one's math knowledge and skills learned over the years.

   This book places considerable emphasis on students learning to read math and to read across the curriculum areas that make use of math in representing and solving problems. The discipline of mathematics has great breadth and depth. Both are a challenge to students, and many students forget much of the math they have learned in the past. This is because of lack of routine use of that math. Math word problems (as well as other types of math problems) can be designed so that students need to draw on the full range of the math they have studied in the past.

6. The problem facilitates students gaining increased knowledge and skills in math modeling and Computational Thinking as an aid to representing and solving problems. (See http://iae-pedia.org/Computational_Thinking.) Calculators, computers, and computerized aids to representing and solving problems are all powerful resources. It takes considerable learning and experience to become skilled in the use of such resources. One indicator of increasing math maturity is increasing knowledge and skill in making use of

computerized tools as an aid to representing and solving math problems and math-related problems.

Our educational system faces a major challenge that is represented by the question, "If a computer can solve or make a major contribution in helping to represent and solve a type of problem we want students to learn about in the curriculum, how should this computer capability be incorporated into the curriculum content?"

Our math education system has made some progress in addressing this question. Thus, inexpensive 6-function calculators are incorporated into some parts of the elementary school curriculum, and more powerful scientific calculators and graphing calculators are often used in the high school math curriculum.

However, there is a growing gap between student understanding of roles of computers in problem solving and the capabilities of computers.

7. The problem is challenging, requiring students to draw on their higher-order cognitive skills. Typically, such a problem requires concentrated effort over a significant period of time. Contrast this with students working on a set of "exercises" designed to give students practice on a new idea they have just learned.

   One indicator of increasing math maturity is increasing ability to solve problems that require sustained effort over an extended period of time. Of course, the meaning of "extended period of time" varies with the overall cognitive development level and attention span of students. The routine incorporation of such problems into the math curriculum is designed to help students move away from the need for instant gratification and towards the deeper type of gratification that comes from successful sustained effort.

You may well want to expand the above list to fit with your particular insights into teaching and learning math. Also, you may want to read the following two Wiki documents:

- Good math lesson plans (http://iae-pedia.org/Good_Math_Lesson_Plans).
- Math project-based learning (http://iae-pedia.org/Math_Project-based_Learning).

Both of these documents consider the issue "goodness" of math lessons and instruction. Math project-based learning (PBL) is especially important because it provides an opportunity for students to create (define) math-related problems and then communicate the results of their work. WordsWorth is a rich source of both problem-based and project-based learning.

Math Maturity Food for Thought. Do you have your students create math word problems? Think about what might be gained by an increased emphasis on this aspect of math education. Students are asked to create problems that are interesting, challenging, and relevant to the math curriculum and applications of math in other disciplines. These problems can be shared with other students, and posted on a bulletin board or on a computer system.

Note that being able to pose a good problem and being able to solve a problem posed by oneself or other are two different things. Researchers in every discipline pose problems that they have not solved. Indeed, they often pose problems that they and others are unable to solve.

This type of assignment can be used as a math project-based learning activity. Have you ever made use of PBL in your math teaching? Think about how PBL might be useful in involving students in activities that will help to increase their levels of math maturity.

Final Remarks

Earlier parts of this book have emphasized that we all routinely deal with word problems in our conversations with other people. We deal with word problems when we tell a machine such as a computer or an ATM to do our bidding. We deal with word problems as we read for understanding.

Word problems—whether they are presented orally, on a computer screen, or in hard copy print—are a routine part of the world in which we live. Because math is such a useful aid to representing and solving problems, many of the word problems that people encounter are math related.

The math curriculum places considerable emphasis on helping students get better at solving word problems. This chapter lists some possible characteristics of good math word problems. You are encouraged to make use of math-related word problems in a manner that will help your students increase their abilities to use their math knowledge and skills to help represent and solve problems in all disciplines.

Activities and Possible Homework Assignments

1. **(For use with students.)** Talk to several children to learn whether they can tell you some general-purpose strategies they use when faced by novel problems. In the process, pay attention to whether the children have vocabulary (such as the word *strategy*) useful in carrying on the conversation and in thinking about how they approach novel problems. Also, focus on problems from many different disciplines—not just math problems or math exercises.

2. **(For use with students.)** Working with a group of students, such as a whole class, determine how many are familiar with Sudoku. If quite a few are familiar with this puzzle, then have the Sudoku-experienced class members teach the game to the others, working in one-on-one or in very small group instruction mode. If few are familiar with the puzzle, teach it to the class. Make use of your Sudoku-knowledgeable students as aides to help the other students as they work on a puzzle. Then debrief this learning experience with the whole class. Direct the conversation so you gain increased insight into students helping students, students being helped by students, and the overall student experience in learning and playing with this puzzle.

   This same type of activity can be done with KenKen. (See Chapter 14 and http://www.kenken.com/.)

3. **(For use with students.) Here is a project for your students.** Identify a school playground or a small park that is familiar to students you work with. Divide your students into groups of three or four. Each group is to figure out how to significantly improve the park or the playground. Resources include up

to $50,000 as well as possible volunteer time contributed by parents and children. A team's completed play is to be presented to the whole group.

One aspect of a plan and presentation is apt to be a careful drawing (to scale) of the area to be renovated and items to be added or removed. Another aspect is a careful budget—which certainly requires some research. A third aspect is estimating the amount of labor to be done, how much can be done by volunteers, and how much the rest will cost. As you can see, there is a lot of math as well as other activities involved in this project.

4. **(For use with students.)** Provide your class with several examples of word problems that are suited to their level of reading and math knowledge and skills. Facilitate a discussion of what makes word problem hard and what makes a word problem easy. Facilitate a discussion of some of the good characteristics of a good word problem. Then, assign your students the task of writing word problems that are good, challenging, and suitable for use in the class.

5. **(A possible homework assignment or discussion topic in a course.)** Select a math textbook that contains quite a few word problems. Find one or two that are sufficiently interdisciplinary so that a student needs to know a reasonable amount about the non-math content area or do some studying of the non-math content area to be able to have a reasonable chance of solving the problem. Use this activity to increase your insight into interdisciplinary uses of math.

6. **(A possible homework assignment or discussion topic in a course.)** Select a math textbook that contains quite a few word problems. Analyze some of these problems using the characteristics of "goodness" discussed in this chapter. Pick out several that you feel rank relatively high and several that you feel rank relatively low. Is there a small set of characteristics of the more highly ranked problems? Is there a common set of missing characteristics of the more lowly ranked problems?

7. **(A possible homework assignment or discussion topic in a course.)** Think about your level of knowledge and skill in using calculators, computers, and computerized tools to help represent and solve math problems. For the latter part of this thinking activity, make a list of tools that you use. Examples might include a calculator, credit card, cell phone, speedometer and odometer, GPS, computerized thermostat, and so on. Then summarize your overall thinking in this activity by analyzing how the math curriculum you teach helps prepare your students for adult life use of calculators, computers, and computerized tools.

Chapter 8: Math Games and Puzzles

> "It's not whether you win or lose, it's how you play the game." (Grantland Rice; American sportswriter; 1880–1945.)
>
> "The reason most kids don't like school is not that the work is too hard, but that it is utterly boring." (Seymour Papert; born in South Africa; MIT mathematician, computer scientist, and educator; 1928–.)

This chapter explores how math-oriented games and puzzles can be used to help improve math education. There is a special focus on improving math maturity.

We are especially interested in activities that engage students and that students find intrinsically interesting and motivating. These characteristics increase the likelihood that children will voluntarily and enthusiastically spend time in these activities.

Introduction

In this chapter, *game* includes electronic and non-electronic games and puzzles. Many games are playable both in a computer mode and a non-computer mode. For example, many solitaire card games and poker games require only a standard 52-card deck. The WordsWorth games presented in Chapter 1 do not require any elaborate or expensive equipment. As you know, however, there are a steadily increasing number of electronic games that are designed to catch and hold the attention of game players.

Mitchell and Savill-Smith (2004) is a British government-funded review of the computer game literature. The following quote from this document offers definitions of *play* and game.

> First, **play**: something one chooses to do as a source of pleasure, which is intensely and utterly absorbing and promotes the formation of social groupings (Prensky 2001, page 112). Fun, in the sense of enjoyment and pleasure, puts us in a relaxed receptive frame of mind for learning. Play, in addition to providing pleasure, increases our involvement, which also helps us learn (Prensky 2001, page 117).
>
> …
>
> Second, a **game**: seen as a subset of both play and fun (Prensky 2001, page 118). A game is recognized as organized play that gives us enjoyment and pleasure (Prensky 2001). Dempsey et al. (1996, page 2) define a game as: …a set of activities involving one or more players. It has goals, constraints, payoffs and consequences. A game is rule-guided and *artificial in some respects.*

Most children who play games, play for fun. Many adults also play games for fun, but some play to make a living or perhaps because they are addicted to a game. For many people, games are attention grabbing and attention holding. They are intrinsically motivating, and they may be addictive. This is an important idea to keep in mind as you explore possible roles of Games-in-Education. Your authors are interested in how games can be used to improve education. At the

same time, we are fully aware that games can damage a person's education and other aspects of their life. For example, it is well known that gambling games have seriously damaged or destroyed many lives!

Research in recent years suggests that media developers (including game developers) are being more and more successful in capturing the time of children. Quoting from Kaiser Family Foundation (1/20/2010):

> A national survey by the Kaiser Family Foundation found that with technology allowing nearly 24-hour media access as children and teens go about their daily lives, the amount of time young people spend with entertainment media has risen dramatically, especially among minority youth. Today, 8-18 year-olds devote an average of 7 hours and 38 minutes (7:38) to using entertainment media across a typical day (more than 53 hours a week). And because they spend so much of that time 'media multitasking' (using more than one medium at a time), they actually manage to pack a total of 10 hours and 45 minutes (10:45) worth of media content into those 7½ hours.

From a formal education, schooling point of view, this situation is a disaster. School is perhaps six hours a day for 180 days a year. That is a total of 1,080 hours per year. However, the media use being measured in the research report averages over 2,500 hours a year. Moreover, the research report does not include time students spent on the phone talking or text messaging.

Marc Prensky

Mark Prensky has been making major contributions to the game world for many years (Prensky 2001, 2002, n.d.). The following is quoted from Prensky (2001), with comments from your authors inserted in square brackets:

> Computer and videogames are potentially the most engaging pastime in the history of mankind. This is due, in my view, to a combination of twelve elements:
>
> 1. Games are a form of fun. That gives us enjoyment and pleasure. [Learning can be fun. But, to achieve a reasonably high level of expertise in an area takes a lot of hard work. Many who are willing to do such hard work find pleasure in it.]
> 2. Games are form of play. That gives us intense and passionate involvement. [Many students never achieve a similar level of passion and involvement in the learning environment provided by schools. Good teachers and good school environments help students to have intense involvement and passion for learning.]
> 3. Games have rules. That gives us structure. [Schools have rules, but they often seem rather arbitrary and not necessarily understandable and relevant to students. Math can be considered to be a game that has precise rules and structure.]
> 4. Games have goals. That gives us motivation. [Learning to set and achieve goals is an important aspect of a good education.]
> 5. Games are interactive. That gives us doing. [In a game, one is interactively involved in solving a problem or accomplishing a task.

Many games involve more than one player, thus introducing the added interactivity with other people.]

6. Games are adaptive. That gives us flow. [Many computer-based games have a wide range of levels of difficulty. The concept of "flow" comes from the work of Mihaly Csikszentmihalyi, and can include being so deeply involved in a game that one shuts out the rest of the world and experiences something like a trance-like state of mind. See http://en.wikipedia.org/wiki/Mihaly_Csikszentmihalyi.]
7. Games have outcomes and feedback. That gives us learning. [Good, timely feedback is very important in learning. In a game, the feedback may come from one's self, other players, observers, and coaches. If the game is computerized, feedback might also come from the computer system.]
8. Games have win states. That gives us ego gratification. [Many of the popular computer-based games have "levels." These can be thought of as intermediate or "progress" goal states. Typically, one can see their progress toward achieving a goal state. For example, in playing the solitaire card game named Klondike, one can count the number of cards moved to the four foundation piles. One can see progress toward getting all 52 cards moved to the foundations. One can keep records of the number of cards played to the foundations over a period of games. And, of course, in competitive games one may win, draw, or lose.]
9. Games have conflict/competition/challenge/opposition. That gives us adrenaline. [Some games are not competitive, and some games are competitive. Some games involve both competition and collaboration. For example, in some types of Dungeons and Dragons games, a team of players work collaboratively, but in competition with other teams.
10. Games have problem solving. That sparks our creativity. [One can learn a great deal about problem solving through playing games. Moursund's (2008) book about games and education uses problem solving as a unifying theme.]
11. Games have interaction. That gives us social groups. [All games have some sort of interaction between a game player and the game itself. This is true even in solitaire games. Of course, there are many games where one's fellow players (opponents and/or partners) are all played by a computer. It is possible to be the only human in a multi-player game.]
12. Games have representation and story. That gives us emotion. [One way to think about this is in terms of a game having a set of rules, and that these rules are an introduction to the story. As you learn to play the game and then play the game, the story of your involvement with the game grows. The playing produces a multi author interactive story. In addition, many games have long and interesting histories. These histories may well include stories about various successful and/or interesting players of the game.]

Math Maturity Food for Thought. Many students think of school as a game in which they compete against the "system" and the teachers. Spend a few minutes thinking about informal and formal education as a game. Which of the Prensky's 12 game descriptors seem to apply to formal schooling, and which clearly do not? What does this type of analysis tell you about our educational system?

The **Math Maturity Food for Thought** activity suggests that games have a number of different values, and that some of them are educational. For example, consider the computer simulations and computerized simulators used to help train astronauts, airplane pilots, car drivers, ship captains, medical students, business students, and so on. These are used because they are both effective and cost effective.

Many different important, real world training or combined training and education learning tasks can be greatly improved by use of computer simulations. If you are a Star Trek fan, you are familiar with the Holodeck (http://en.wikipedia.org/wiki/Holodeck). Computer technology is gradually catching up with such science fiction, thus making it possible to immerse learners in highly realistic, interactive simulations of considerable educational value.

These types of training and education activities are backed up by substantial research—they work! However, the real-world versions of such computer simulations and computer-based simulators tend to be relatively expensive. One reason for this is that very careful consideration needs to be given to transfer of learning from the simulation to the environment being simulated. Ideally, a person using a high quality airplane pilot flight simulator cannot readily tell the difference between being in such a simulator and actually flying an airplane.

Some Research Considerations

Transfer of learning is certainly one of the most important ideas in education. We want the learning that students obtain in school to transfer to both non-school settings and to future schooling settings. In some sense, schooling can be thought of as immersing students in a type of simulation that is supposed to prepare them for life outside of the school classroom and for what they will encounter in their future formal and informal education and training.

We want students to learn more than just the content being covered in a course. We want students gain confidence in their knowledge and skills, develop a positive attitude toward learning and using their education, value and respect the learning of others, learn to work in team settings, and so on.

Education and Training

There are other important issues, such as efficiency of learning. How fast and how well does a student learn, and how well does the learning last? When is "training" rather than "education" a good approach? In brief summary, the research indicates that if you want a student to learn to solve a specific type of problem or accomplish a specific type of task, teach directly toward these goals (Kirschner, Swelner, and Clark, 2006). Thus, if you want a student to learn to solve linear equations, directly teach how to solve linear equations. As students gain in math maturity, they can provide more of the needed math-learning guidance to themselves.

The Kirschner, Swelner, and Clark (2006) article leads to the following suggestions to teachers and parents:

1. If there is a type of problem or task that students will frequently encounter and it is deemed important that they gain skill, speed, and accuracy in handing the problem or task, strongly guided instruction is a good approach.
2. Teach in a manner that moves students in the direction of being able to provide their own "internal" guidance and to be intrinsically motivated. Help them learn to learn on their own and to take an increasing level of responsibility for their own learning.
3. Help students learn to solve novel problems and accomplish novel tasks in areas and topics where they have not received strongly guided instruction. This includes directly teaching for transfer of learning.

You are familiar with the terms training and education. Training prepares learners to deal with an anticipated and clearly defined set of problems and tasks. Education prepares students to deal with unanticipated, novel, and not necessarily clearly defined problems and tasks.

We live in a world that is complex and growing more complex, and in which the totality of accumulated data, information, and knowledge is growing quite rapidly. We all face the challenges of information overload. See http://iae-pedia.org/Information_Underload_and_Overload. Students need an education that prepares them for being responsible, productive, and caring adults in this changing world.

Thus, our overall educational system and each individual student face the complex issue of training versus education. We see this in a university that has a College of Liberal Arts and a number of Professional Schools such as Architecture, Business, Engineering, Journalism, Law, Music, and so on. We see this in secondary schools that offer general coursework preparing students for college, and more specific coursework preparing students to directly enter the job market.

Math Maturity Food for Thought. The previous chapter discussed problem solving, with an emphasis on math word problems. Think about the following situation. A number of books and articles have been written teaching students to solve various types of word problems that are frequently used in standard textbooks as well as on state and national tests. This is often called "teaching to the test." In essence, the teaching materials assume that the goal is to train students in a manner so they will do better at solving the anticipated types of word problems. Thus, for example, students might be trained to solve the type of problem in which two vehicles are approaching each other and a bird is flying back and forth from one vehicle to the other. On a state or national test, being facile at solving variations on this problem situation may well help a student to quickly and correctly solve that type of problem. One additional correct answer can be quite important to the student. Reflect on this situation, and analyze your arguments for and against this type of math training. Take into consideration how this type of training contributes to (or, fails to contribute to) an increased level of math maturity.

Puzzles

A puzzle is a type of game. To better under the purpose of this section, think about some popular puzzles such as crossword puzzles, jigsaw puzzles, Sudoku puzzles, KenKen puzzles,

and logic puzzles (often called brainteasers). In every case, the puzzle-solver's goal is to solve a particular *mentally challenging* problem or accomplish a particular mentally challenging task.

Many people are hooked on—in some sense, addicted to—certain types of puzzles. For example, some people cannot start the day without spending time on the crossword or Sudoku in their morning newspaper. In some sense, they have a type of addiction to crossword and/or Sudoku puzzles. The fun is in meeting the challenge of the puzzle—making some or a lot of progress in completing the puzzle.

Crossword puzzles draw upon one's general knowledge, recall of words defined or suggested by short definitions or pieces of information, spelling, and recognizing a word from a few letters of the word. Through study and practice, a person learns some useful strategies and can make considerable gains in crossword puzzle-solving expertise. Doing a crossword puzzle is like doing a certain type of brain exercise. In recent years, research has provided some evidence that such brain exercises help stave off the dementia and Alzheimer's disease that are so common in old people.

From an educational point of view, it is clear that solving crossword puzzles helps to maintain and improve one's vocabulary, spelling skills, and knowledge of many miscellaneous tidbits of information. Solving crossword puzzles tends to contribute to one's self esteem. For many people, their expertise in solving crossword puzzles plays a role in their social interaction with other people.

A recent Internet search of *math puzzles* produced near 4.5 million hits. For example, see Figure 8.1, one of many different math puzzles available at http://images.google.com/images?hl=en&q=math+puzzles&rlz=1B5GGGL_enUS316US317&um=1&ie=UTF-8&ei=7Oa4S9SkJ4zasQOYlZHpDA&sa=X&oi=image_result_group&ct=title&resnum=5&ved=0CDAQsAQwBA.

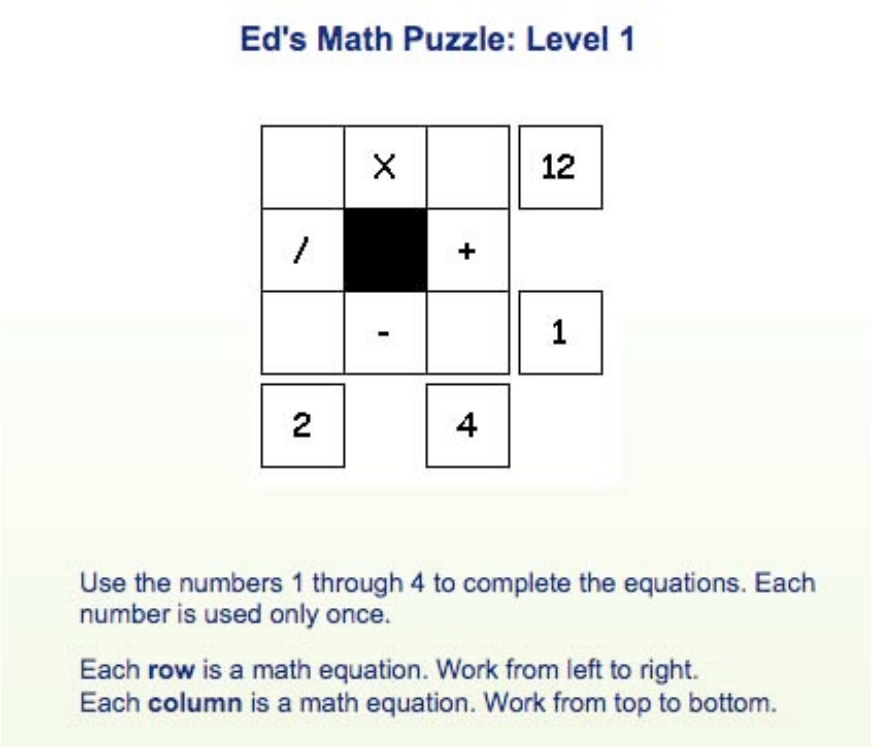


Figure 8.1. A simple math (arithmetic) puzzle.

Competitive, Independent, and Cooperative

A game can be analyzed from a point of view of its:

- Cooperation and collaboration (non competitive)
- Independence (non cooperative, non competitive) ,
- Competition leading to the determination of winners and losers.

Roughly speaking, some people tend to be very competitive, some tend to be non competitive, and some tend to be very cooperative/collaborative. However, a person might fall into one of these three categories in certain situations, and in a different category in other situations. For example, think about a parent being extremely cooperative/collaborative when playing a board game with his or her young children, independent when playing a solitaire card game or Tetris, and perhaps quite competitive in playing poker or bridge with his or her adult friends.

Early developers of computerized games found that the types of games they developed tended to be quite appealing to males and not to females. Research indicates that on average, women tend toward the cooperative/collaborative end of the scale while men tend toward the other end of the scale. In recent years, a number of computer game developers have placed considerable emphasis on developing games that are appealing to females.

The puzzle video game Tetris created in 1984, along with many variations of this game, is particularly attractive to girls and women. (See http://en.wikipedia.org/wiki/Tetris.) It typically is played as a solitaire, non-competitive game. It is playable at many different levels of difficulty. It

is a game in which one can improve their skill through practice. It involves quick recognition of certain geometric shapes and good hand-eye coordination.

Precise Vocabulary and Notation

In many games it is possible (and, often desirable) to keep good records of the moves or activities in the game. This allows one to replay the game and to carefully analyze various aspects of the play.

Chess provides a good example. Figure 8.2 shows a chessboard. Notice that the columns (the files) of the 8 x 8 board are lettered a, b, … h, and the rows (the ranks) are numbered 1, 2, … 8. In chess, the person playing the White pieces always moves first. The lettering and numbering notation used to identify the spaces on the board is convenient and natural from the point of view of the person playing the White pieces.

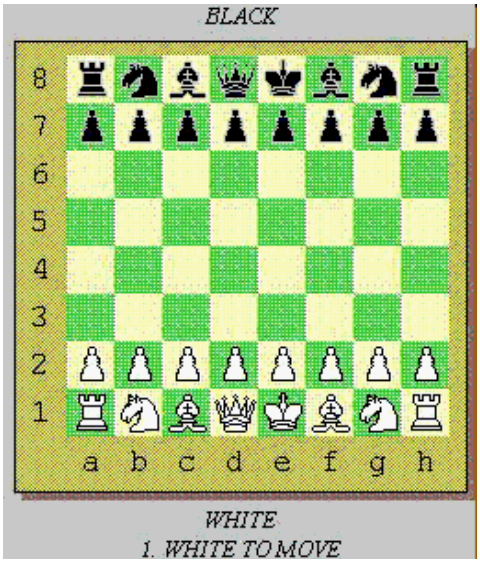


Figure 8.2. Lettering of columns and numbering of rows.

The names of the pieces are abbreviated as follows: K=King, Q=Queen, R=Rook, B=Bishop, N=Knight, and P=Pawn. This board coordinate system and the piece name abbreviations make it quite easy to record all of the moves in a game. For example, here are the first few moves of a game. The listing indicates that White's Bishop captures Black's Knight on White's fourth move.

1. Pe4 Pe5
2. Nf3 Nc6
3. Bb5 Pa6
4. BxN

This, and other notational systems that are widely used in chess, allow players to record precisely the moves in a game (Calvin, n.d.). Students who take a serious interest in chess while they are quite young learn this algebraic notation well before they learn about algebra or coordinate systems in their math classes.

Keeping a detailed record of one's chess games and studying both one's own and other people's games is a strategy used to improve one's level of expertise in chess. Is this type of strategy applicable to other games? Is it applicable to non-game learning and problem-solving situations? Of course it is. So, let's give this strategy a name. Let's call it the *record one's moves strategy*. This is what a researcher does when conducting research in any field. Details of the research need to be precisely recorded so that the researcher and/or others can duplicate the experiment. Thus, it should be part of the repertoire of high-road transferable problem-solving strategies that you and your students routinely draw upon.

Math Maturity Food for Thought. Reread the previous two paragraphs. Through learning chess notation and how to make a written record of a chess game, chess-playing students learn:

1. How to read, write, and make effective use of chess notation. Chess notation is an important aspect of the language of chess.
2. To make use of a two-dimensional coordinate system in a situation that is relevant to their interests.
3. A fundamental aspect of research—keeping a careful record of one's work so that oneself and others can review it.

Analyze items 2 and 3 in terms of possible transfer of learning. Thus, for example, think about how item 3 relates to the teaching, learning, and doing of science.

The *record one's moves strategy* helps to explain why each discipline tends to have some special notation and definitions of terms that are unique to the discipline. It is absolutely essential that people working in a discipline be able to accurately record the work they are doing so that it can be precisely communicated to others and to themselves. A novice in a discipline needs to learn the precise notation and vocabulary in order to take advantage of the accumulated knowledge in the discipline. That is, part of learning a discipline is to learn to read (for understanding) in the content area of the discipline.

What Constitutes a Good Educational Game?

This book focuses on uses of math-oriented educational games and math-oriented word problems that are designed and used in a manner to help improve math maturity. Chapter 2 lists quite a number of possible components of math maturity. You can personally evaluate a math-oriented educational game from several points of view:

1. Is it clear in your mind and/or in the minds of people whose judgments you trust, that the game has considerable potential to help increase the player's levels of math maturity? If a parent or school administrator visits your classroom while students are playing the game, will you feel comfortable in explaining the game's educational values to the visitors?
2. Is the math-related problem solving content consistent with and supportive of the math education goals that you hold in mind as you teach math?
3. Do you have the needed level of understanding of the game as well as teaching knowledge and skills to help students learn the game and to effectively integrate it into your math curriculum? A different way of dealing with this question is to decide whether you are willing to put in the time and

effort to gain the necessary game, teaching, and classroom management knowledge and skills.

4. Are you able to set clear educational goals that you will hold in mind as you make use of the game? Remember, the intent is to make use of the game to improve the quality of education that your students are obtaining. If you cannot see and then teach toward clear educational goals related to use of the game, then you should give the game a poor rating and not attempt to use it in your teaching.
5. Does the game have the flexibility and varying levels of difficulty so that it can meet the needs of the wide range of students you are working with?
6. Does the game have a good potential for transfer of learning to important other aspects of the curriculum you are teaching? A game may be a close simulation to a problem-solving or task-accomplishing situation that you want your students to learn to effectively deal with. Or, a game may seem to you to be quite unrelated to your teaching goals. Moursund's (2008) book on using games in education has a major emphasis on transfer of learning from games.
7. Can students get started in playing the game relatively easily and quickly, and then learn more by playing and through instruction based on situations that occur in playing the game? That is, is the game one in which there is easy entry but a high ceiling?

Math Maturity Food for Thought. Select a game that you are familiar with and that you feel might be suitable for use with students that you teach. Think about which of the above criteria are satisfied by this game. Also think about criteria that you feel should be added to the list.

General educational research gives us some insight into training (preparing to deal with a relatively well defined type of problem or task) and education (preparing to deal with less clearly defined, more novel problems and tasks). We know about competitive and non-competitive games. We know about games that promote social interaction and solitaire games. And, of course, all forms of informal and formal training and education can be analyzed from the point of view of transfer of learning over time and space.

Thus, we can analyze any game from these and other points of view. These types of analyzes can give us some insights into what constitutes a good game. Without even starting such analysis, we know that a particular game may well score high from one point of view and low from another. Moreover, one game might be well suited to some students and not for other students.

Of course, these are not profound insights. Our whole educational system, with its wide variety of students, subject areas, curriculum designs, textbooks, media, and so on has similar characteristics.

Our goal in this section is to provide you with a little help as you select games that will have significant educational value when used by children and students (home and/or school) that you care about.

Here are a few ideas that your authors have had in mind as they have decided on what games to emphasize in this book.

1. We want games that can be played at home or school and that use inexpensive materials. For example, consider games based on each player or small group of players having a complete 52-card deck of cards versus each player or small group of players having a few pairs of dice. The logistics of a large number of decks of cards (for example, continually having to check to see that each deck is a full deck with no duplicate cards and no missing cards) are large relative to having a box of dice or boxes of various colors or types of dice. Moreover, if special decks of cards are needed—which is common in many game and education settings—both the logistics and the cost can be a challenge. Similarly, consider games that require special scoring or record keeping preprinted sheets or other materials that are used and discarded, are easily broken, or wear out relatively quickly. Your authors have tended to avoid such games in this book.
2. Generally speaking, any game can be played for educational purposes. Your authors are interested in games that are designed for a combination of educational and entertainment value, and where the educational values are clear. When such games are used in situations (environments) that are encouraging and supporting learning, learning is apt to occur. Thus, your authors are interested in games where the learning goals are clear to and supported by the adults providing students with access to the game, doing the initial teaching of how to play the game, and providing general supervision of the overall operation.
3. Games vary considerably in how long they take to play. Your authors are particularly interested in games that will fit well into typical school schedules. Such games need to be easy to set up and get started, relatively fast paced and short, and easy to end and take down.
4. We are also interested in games that can hold the attention span of the players over a significant length of time—for example as the game is used daily or weekly over a long period of time.
5. We are interested in games that can be played alone or in small groups, and that come in a broad range of difficulty levels. For example, the easiest versions of WordsWorth Plus can be played by first grade students, while there are versions of WordsWorth that are challenging to high school students.
6. We are interested in games that students find sufficiently interesting that they will tell their parents, siblings, and friends about their in-class game-playing experiences and what they learned from playing the game. We feel that there should be explicit instruction on what students are learning by playing the game.

Final Remarks

There are a huge number of games that share the characteristics of being both educational and entertaining. Some require very little investment in materials and in student time to get

started, but have a very high ceiling level. Such games help a learner learn about learning and the effects of dedicated study and practice over an extended period of time.

Some involve developing and using strategies that are applicable to game playing and other problem solving in many different areas. Such games can help a student learn about and get better at transfer of learning.

Many games involve two or more players. Such games can help students learn about competition and teamwork, and can help students develop certain types of social skills.

Computers have added a new dimension to the gaming world. Today's children are growing up in a world that competes fiercely for their time and energy. Game developers strive to develop games that will catch and hold a player's attention—games that have addictive-like characteristics. A flip side of this situation is that with appropriate help and supervision, children can learn about addiction to games and how to deal with such addiction tendencies. Such learning environments can also be used to help students learn about other types of addictions.

Activities and Possible Homework Assignments

1. **(For use with students.)** What are some games today's students find fun to play? Ask your students to help you make a list of games they have played and enjoyed. As the list is being created, divide its items into five categories:

    a. Board games, card games, and other types of non-electronic games that are not physical sports games.

    b. Electronic games involving one or a small number of players.

    c. Electronic games involving a large number of players. These are called massively multiplayer games. See http://en.wikipedia.org/wiki/Massively_multiplayer_online_game. The online game *World of Warcraft* has many millions of players.

    d. Physical sports games such as basketball and skipping rope (jump rope).

    e. Other (not fitting easily into any of the above categories. (Twister would be an example. See http://en.wikipedia.org/wiki/Twister_%28game%29.)

    Use this activity to promote a discussion about whether a game can fit into more than one category, what a game is and is not, whether a puzzle is a game, what makes a game fun, whether and how a game be fun for one person and not for another, and so on.

2. **(For use with students.)** Engage students in a discussion about what they have learned and other ways in which they have benefited by playing the various types of games on the list generated in (1) above. Make sure the discussion includes topics such as intrinsic motivation as well as learning that leads to getting better at the game.

3. **(For use with students.)** Engage your students in making a list of games that they consider to be educational and relevant to the coursework they are taking from you. Have the students decide on one or more games that they would like to try out during class time. Then, just do it! Have the students bring the

necessary materials if you do not already have them available, and have the students who know a game that has been selected teach it to the rest of the class.

4. **(A possible homework assignment or discussion topic in a course.)** Identify some board games and/or card games that you remember playing as a child. What made them fun (or, not fun) to play? How did these games contribute to your education?

5. **(A possible homework assignment or discussion topic in a course.)** In playing board games and/or card games, what is your preference on a scale from non-competitive games to highly competitive games? Does your preference affect your choice of games and related teaching activities used in your teaching? For example, if you are a quite competitive person, do you tend to emphasize competitiveness in your teaching?

Chapter 9: Dice, Coins, and Chance

> "God does not play dice with the universe." (Albert Einstein; German-born theoretical physicist and 1921 Nobel Prize winner; 1879–1955.) Einstein's famous quotation was not about his speculations concerning the gambling propensities of God. Rather, it was an expression of his dissatisfaction with the apparently probabilistic description of nature embodied by the quantum theory.

This chapter is the dividing line between the book content that focuses on general ideas and the book content that focuses on specific games. Many of the games given in subsequent chapters include some element of randomness that is produced by rolling dice, flipping coins, drawing tiles that are face down, and so on.

Dice are used in some games of chance and in a variety of other games where one wants to generate a random number. Perhaps you are familiar with Craps and Monopoly. Craps is a dice-based gambling game, while Monopoly is a board game that uses a pair of 6-faced dice (which we abbreviate as 2D6) as illustrated in Figure 9.1.



Figure 9.1. Two 6-faced dice (2D6)

The oldest known dice are part of a 5,000 year old Backgammon game discovered at an archeological dig in south-eastern Iran. See http://en.wikipedia.org/wiki/Dice.

This chapter provides a brief introduction to randomness and probability. Probability is a part of the National Council of Teachers of Mathematics math standards and is an important aspect of our everyday lives. Ideas about probability and chance are touched on in the lower grades and more explicitly taught in the higher grades in math education.

Cubical Dice

The most commonly used dice are cube. Dice come in a variety of colors and sizes. The six faces of a D6 each have a different number of pips (spots) in the range integer [1, 6]. The math notation integer [1, 6] means the integers 1, 2, 3, 4, 5, and 6. The sum of the pips on opposite faces of a die is 7.

When a D6 is rolled on a hard level surface, it will come to rest with one face up and one face down. The face that is up is called the *outcome* of the roll. The number of pips (or spots) identifies the outcome. The outcome from rolling 1D6 is one of the integers [1, 6].

A *true die* is one for which each face is equally likely to be the outcome when the die is rolled. Thus, the roll of a true D6 produces a random integer in the range [1, 6] with each of these 6 different outcomes being equally likely.

Of course, a D6 can be loaded so that the six possible outcomes are not equally likely. Moreover, small variations in the manufacturing of dice may produce dice that are not true. See http://en.wikipedia.org/wiki/Dice#Crooked_dice.

Children are often introduced to dice as they learn to play board games in which the roll of 1D6 or 2D6 determines how far one's piece is moved as it traverses around a board. A young child's first board game might use 1D6. Initially, the young child counts the pips on a die. Eventually, however, the child memorizes the various pip patterns and the corresponding integers. A human brain is good at pattern recognition, learning new patterns, and developing automaticity in this endeavor.

When 2D6 are used, the child is initially faced by the task of counting up to 12 pips. Eventually this task is simplified to recognizing the pip patterns on each die to give two integers in the range [1, 6] and adding the two integers. With still more practice, a child learns to do this addition quickly and accurately with little conscious effort. Indeed, perhaps the child memorizes (completely automates reading) the dice patterns that produce the various outcomes integer [2, 12].

Notice the amount of learning that is going on as a child develops fluency in making board game moves through use of 2D6. This learning occurs in an environment of play—with the players being peers, parents, and others. The child is immersed in a fun social environment, and the learning is fun.

This simple example gives us considerable insight into learning and the human brain. With sufficient practice, the problem of determining the outcome from rolling 2D6 can become completely automatic—it is accomplished quickly, accurately, with little conscious effort.

Math Maturity Food for Thought. This rote learning process can be repeated with learning to solve other frequently occurring problems. If life in our society involved dealing with only a modest number of problems, informal and formal education might well consist of a rote memory approach to mental automation in solving each of the problems. However, life is not so simple!

Spend some time thinking about a rote memory approach to problem solving versus an understanding approach to problem solving. How do you—as a learner and/or as a teacher—decide when rote memorization is an appropriate learning approach versus when other approaches such as learning with understanding, learning to figure it out, learning to "look it up," and so on are more appropriate? Note that each student is different from all other students. How do we decide what **all** students should be required to commit to rote memory? Remember, children vary considerably in how easily they memorize and how long they can remember what they have memorized.

Dice Made from Regular Platonic Solids

A D6 is cube-shaped. A cube is one of the five regular Platonic solids shown in Figure 9.2.



Figure 9.2. The five regular Platonic solids.

Notice that the 4 faces of the tetrahedron are all congruent equilateral triangles. The 6 faces of a cube (hexahedron) are all congruent squares. The 8 faces of an octahedron are all congruent equilateral triangles. The 12 faces of a dodecahedron are all congruent pentagons. The 20 faces of an icosahedron are all congruent equilateral triangles. The idea of congruence is an important aspect of geometry.

All of the vertices in a Platonic solid have the same number of edges meeting at the vertex. Thus, for example, in a cube each vertex is formed by the intersection of 3 edges, and in an octahedron each vertex is formed by the intersection of 4 edges.

Also, notice that all of the Platonic solids are convex polyhedrons. *Convex* means that every line drawn between two points of the polyhedron lies entirely in the polyhedron.

A mathematically oriented person might well ask the question, "Are there any other regular Platonic solids?" Asking such questions is an indication of growing mathematical maturity. It turns out that there are only 5 Platonic solids.

The picture in Figure 9.2 and the following quote are from http://en.wikipedia.org/wiki/Platonic_solid.

> The Platonic solids have been known since antiquity. Ornamented models of them can be found among the carved stone balls created by the late Neolithic people of Scotland at least 1000 years before Plato (Atiyah and Sutcliffe 2003). **Dice go back to the dawn of civilization with shapes that augured formal charting of Platonic solids.**
>
> The ancient Greeks studied the Platonic solids extensively. Some sources (such as Proclus) credit Pythagoras with their discovery. [Bold added for emphasis.]

It is easy to see how the Platonic solids can be used to make dice of 4 faces, 6 faces, 8 faces, 12 faces, and 20 faces. Remember, a die is called fair if each of its outcomes is equally likely in a roll of the die. Thus, the roll of a fair D6 produces a random integer in the range [1, 6]. Dice with a larger number of faces, such as the D8, D12, and D20 constructed from Platonic solids typically have numerals rather than pips on their faces. Rolling a fair D20 with faces numbered 1–20 produces a random integer in the range [1, 20].

D4 is a tetrahedron with four faces. When rolled, it has no apparent top face to read. The faces are numbered in a manner to alleviate this problem situation. The number rolled is the one that is right side up after the D4 is rolled. The outcome of the D4 shown in Figure 9.3 is 4.



Figure 9.3. Dice with numbered faces made from Platonic solids.

Math Maturity Food for Thought. Notice the various math words used in the above section. Children playing games that make use of various types of dice might well be learning quite a bit of the math vocabulary from this activity. What are your thoughts about children learning to describe a D6 as a cube whose 6 faces are congruent squares, and that the roll of a D6 produces a random number outcome in the range of 1 to 6 inclusive?

One of the challenges in learning math is to learn the written and oral language of mathematics. Words such as equilateral triangle, polyhedron, and Platonic solid have precise meanings in math. Sometimes math makes use of names of people, such as Plato, Euclid, and Pythagoras in naming math ideas. You have encountered such naming conventions in other disciplines such as science, where we have Newton's Laws of Motion, Einstein's Theory of Relativity, and electrical power measured in watts (after inventor James Watt).

Such use of names is one way to capture a bit of the history of a discipline. However, without knowledge of the specific history of the person, the names just become rather complex identifiers with no attached meaning. Think about what you know about the various people named above, and what you want your students to know about them. Remember, math and science are human endeavors.

Other Fair Dice

It is possible to make fair dice from non-Platonic convex polyhedrons. To understand this, first take a careful look at an octahedron. Each of its eight faces is an equilateral triangle. Visualize an octahedron as two 4-sided pyramids, joined at their bases.

Next, visualize a pyramid with five congruent triangular faces and a regular pentagon base. If two pyramids of this shape and size are joined at their bases, the result is a convex polyhedron with 10 congruent triangular faces. Number the faces 0 through 9 (as in Figure 9.4) or 1 through 10 and you have a D10.



Figure 9.4 A collection of D10.

This same pyramid process can be used to make a D100 or a D1000. However, think about the size that a D1000 would need to be so that each of the 1,000 numerals on its faces is easily read.

If you want to generate a random integer in the range [0, 99], you could use a D100. However, it will likely be more convenient to use a D10 with faces numbered [0, 9]. Roll the

D10 to produce the tens digit and then roll it again to produce the ones digit. Or use D10s in two colors, one for the tens digit and the other for the ones digit.

If you want a random integer in the range [1, 100] instead of [0, 99], you can add 1 to the results of producing a random integer in the range [0, 99] or use a roll of 0 to represent 100.

Two-faced Dice (D2)

Suppose that you want to generate a random integer in the range [1, 2]. One approach would be to use 1D6 and agree that a roll of 1, 2, or 3 is counted as a result of 1, while a roll of 4, 5, or 6 is counted as a roll of 2. Or, use odd for 1 and even for 2.

Still another approach would be to use a coin, with H standing for 1 and T standing for 2. That is, a coin can be thought of as a two-faced die (a D2). Of course, it is possible that a flipped coin might end up standing on its edge rather than displaying just one face! Your authors have never seen this happen.

If you flip a true coin, the chance of getting heads is exactly the same as the chance of getting tails. We say that the chance of each is1/2, or that the probability of each is 1/2. Indeed, that is what is meant by a true coin.

However, an actual coin, such as a penny or a quarter, is likely not a true coin. You can think of an actual coin as a physical model of a true coin. Usually the model is close enough to being a true coin that we typically assume it is true when we flip a coin in order to make a decision.

Math Maturity Food for Thought. Have you ever flipped a coin and had it land standing on its edge? Suppose coins were thicker. Would this increase the chances of a flipped coin landing on its edge? An inquisitive student might ask such a question. In answering this question, it might help you to think of a cylinder, with its length several times its diameter. When such a cylinder is flipped, it is unlikely to land on one of its ends (that is, on one of its circular faces). For example, think of a cylinder formed by taping two dozen pennies or circular plastic chips together.

Doing the type of thinking used in noting that a coin is a cylinder and then thinking about cylinders of various shapes is an indication of math maturity. Part of doing math—indeed, part of any discipline—is creating examples and counterexamples. Think about how you help your students gain in their abilities to create examples and counter examples in math and other disciplines.

Intuition About Dice Rolls

When we roll a D6, we assume that the six outcomes integer [1, 6] are equally likely. This assumption "feels" right—it seems intuitively obvious to most people.

You might want to have your whole class work together to explore this situation. Each student or small team of students is given a D6, asked to roll it a large number of times (such as 50 or 100) and record the number of times it comes up 1, 2, 3, 4, 5, or 6. Each student or small team can analyze their individual results to see if they are "sort of" evenly distributed in the range integer [1, 6]. The analysis might include finding the mean and making a distribution table or graph. The results from the whole class can be combined and analyzed to see if they are "sort

of" evenly distributed in the range integer [1, 6]. Such individual and whole class activities help to develop a student's dice-rolling intuition.

Let's explore the idea that something is *intuitively obvious* a little further. A roll of 2D6 produces two integers that sum to an integer in the range [2, 12]. From the intuitively obvious outcomes from coin tossing and 1D6 rolling, it may seem intuitive obvious that each of the 11 outcomes from rolling 2D6 is equally likely, with each having a probability (chance) of 1/11 of occurring.

However, this intuitive guess is incorrect. Figure 9.5 is table of the probabilities of each of the outcomes in rolling true 2D6. (More explanation of this table is given in the next section.)

Outcome	2	3	4	5	6	7	8	9	10	11	12
Probability	1/36	2/36	3/36	4/36	5/36	6/36	5/36	4/36	3/36	2/36	1/36

Figure 9.5. Outcomes and probabilities in rolling true 2D6.

From the given probabilities, we can see that rolling a 7 is 6 times as likely as rolling a 2 or a 12. The probabilities are determined by careful analysis of rolling a pair of dice. To help in this thinking and counting process, imagine that one die is red and one is green. That allows us to see that the outcome of a red 3 and a green 4 is different from the outcome of a red 4 and a green 3. Although their totals are both 7, the two outcomes are different. This counting process is discussed more in the next section.

Math Maturity Food for Thought. Perhaps you know, understand, and can explain the probabilities given in Figure 9.5. And, perhaps you don't. Think about what difference knowing or not knowing might make for a teacher, parent, board game player, or gambler. Think about situations in your life where understanding probabilities (chances, odds) of an outcome are important to you. Present some arguments for and against requiring the study of probability as part of the math curriculum for all students, beginning at the lower grades.

Have you ever said or thought, "I have an intuition that …" If so, think of some examples in which your intuition was correct and examples in which your intuition was incorrect. Through study and practice in a discipline such as math, one can develop and improve their intuition in that discipline. Such developing and improving math intuition is an indicator of growing math maturity.

Analyzing a Sequence of Dice Rolls or Coin Flips

For many people, their intuition about random events tends to break down when considering a sequence of events. Rolling 2D6 is somewhat like rolling a first D6 and then rolling a second D6—that is, rolling a short sequence of D6s. Flipping a coin five times produces a sequence of outcomes, and the sequence is of length 5. This process has some similarities with tossing five coins simultaneously.

Figure 9.6 is a table of outcomes from rolling a first D6, a second D6, and then calculating their total. The table lists all 36 different possible outcomes and their totals. To save space, the 36 outcomes have been arranged in two groups of 18 each, and placed side by side in the figure.

Study the table for a while to convince yourself that there are indeed 36 possible outcomes. For example, notice that while there is only one way to get a total of 2 (each die must be 1), there are six different ways to get a 7 because 1 + 6 = 7, 2 + 5 = 7, 3 + 4 = 7, 4 + 3 = 7, 5 + 2 = 7, and 6 + 1 = 7. This will help you to understand where the numbers came from in Figure 9.5. Systematic counting of this sort is an important part of the study of probability and is a part of the field of math called combinatorics. See http://en.wikipedia.org/wiki/Combinatorics.

Outcome of first D6	Outcome of second D6	Total
1	1	2
2	1	3
3	1	4
4	1	5
5	1	6
6	1	7
1	2	3
2	2	4
3	2	5
4	2	6
5	2	7
6	2	8
1	3	4
2	3	5
3	3	6
4	3	7
5	3	8
6	3	9

Outcome of first D6	Outcome of second D6	Total
1	4	5
2	4	6
3	4	7
4	4	8
5	4	9
6	4	10
1	5	6
2	5	7
3	5	8
4	5	9
5	5	10
6	5	11
1	6	7
2	6	8
3	6	9
4	6	10
5	6	11
6	6	12

Figure 9.6. Table of all possible outcomes from rolling a D6 sequence of length 2.

The table in Figure 9.7 is a more compact aid to listing and counting outcomes from rolling a D6 sequence of length 2. The top row and the left column each list all possible outcomes from rolling a D6. The shaded 6 x 6 remainder of the table entries contain the sums from one item in the top row and one item in the left column. You can see that all 36 possible outcomes from rolling a first die and then a second die are listed. Use this table to count theoretical frequencies of sums of 2D6. For example, how many 7s? Count them. There are six 7s.

	1	**2**	**3**	**4**	**5**	**6**
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

Figure 9.7. All possible outcome sums from rolling a length 2 sequence of D6 (rolling 2D6).

The graph in Figure 9.8 is a visually useful way of representing the data given in Figure 9.7.

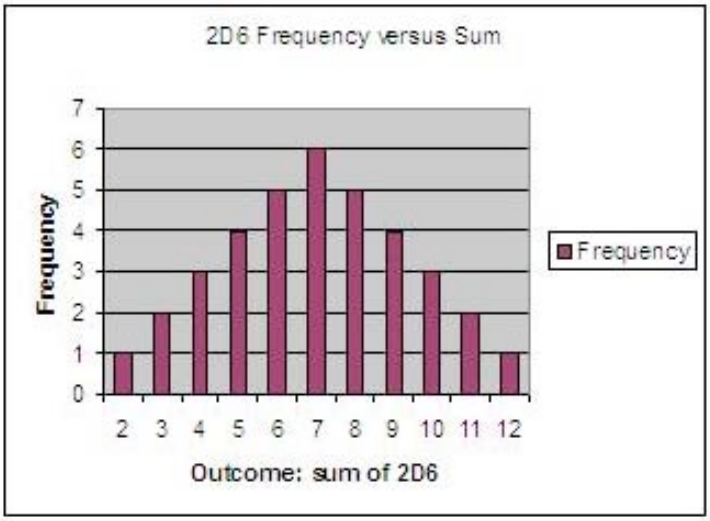


Figure 9.8. Graph of data in Figure 9.6 or 9.7.

Math Maturity Food for Thought. Figures 9.6, 9.7, and 9.8 are all based on the same data. As you attempt to use these three different representations, you might decide that one is more easily used than another. For example, in the table shown in Figure 9.7, it is easy to see the diagonal patterns, such as the 7s along the diagonal from lower left to upper right and to see that there are 6 of these 7s. Contrast that with the greater challenge of counting all of the 7s in the table given in Figure 9.6. The graph in Figure 9.8 presents a picture of the distribution of the sum outcomes.

You know that there are many different ways to organize or represent data. Some ways are "better" (easier to use, more revealing of patterns) than others. This is an important aspect of learning and using math. For example, try to visualize in your mind's eye some possible meanings or representations of the number 144. Perhaps you visualize a quite long line—maybe 144 people standing in line to get into a performance. Or, perhaps you visualize a 12 x

12 square of people at an event sitting in 12 rows each of length 12. A person interested in woodworking might visualize this as one board foot (144 cubic inches) of lumber. A person familiar with base ten blocks used as math manipulatives might visualize 144 in terms of a flat, 4 rods, and 4 cubes. Did you know that 12 dozen is a gross?

One indication of increasing math maturity is an increasing ability to visualize and represent data in a form that is meaningful and understandable to oneself and to communicate such representations to others. How do you help your math students learn to develop multiple representations of data?

Coins and Dice Have No Memory

Suppose that you are watching a coin-flipping game and you observe that H has been flipped 10 times in a row. Do you think to yourself: "Hmm. I am willing to bet that that the next flip will be a tails. It just seems very unlikely that a heads will occur, because I know the heads and tails must balance each other out over time."

This is an example in which one's intuition tends to be faulty. Here are two ideas to think about:

- Suppose that the coin is a true coin. How can the coin possibly know and remember what it has done in the past? A coin doesn't have a memory! By the definition of *true coin*, the next roll has a 1/2 chance of being H and 1/2 chance of being T. The two outcomes are equally likely.
- Suppose you have a suspicion that the coin might not be true. The long sequence of H outcomes provides some evidence (however, not a proof) that the coin is biased toward producing more heads than tails. Since a coin (a true coin or a biased coin) has no memory of what has occurred in the past, the best bet is that H will be the next outcome.

The combination of these two points of view likely leads to betting that H will be the next outcome. If you find a "sucker" who is willing to give you better than even odds that T will occur, you should be inclined toward making such a bet. However, keep in mind that the coin flipper may have been using a biased coin that produces H on every flip, and by a little slight of hand does a quick change to a coin that looks the same, but is biased to produce a T on every flip. You have probably heard the statement, "There is a sucker born every minute." In this situation, who is the sucker?

Computerized Coin Flippers and Dice Rollers

Many students find it is fun to do coin flipping and dice rolling experiments in which they do a large number of rolls or flips, record the results, and then analyze the results. After a while, however, such experiments can become tedious.

There are virtual-coin and virtual-dice programs available free on the Web. With them, one can carry out virtual experiments where the computer does most of the work.

Here are two *virtual coin flippers* available for free use on the Web:

- White, Ken (n.d.). Flip one or more Canadian pennies or dimes. Retrieved 4/28/2010 from http://shazam.econ.ubc.ca/flip/.

- Shodor (n.d.). Coin flipper. Retrieved 4/28/2010 from http://www.shodor.org/interactivate/activities/Coin/.

We used the Shodor coin flipper to flip 100 coins, 20 times. The numbers of heads produced were:

51, 53, 51, 45, 49, 47, 57, 49, 58, 54, 54, 48, 58, 51, 53, 46, 51, 52, 49, 55

We repeated this experiment and produced the results:

49, 56, 55, 50, 55, 49, 51, 51, 56, 54, 46, 56, 47, 56, 40, 52, 51, 63, 45, 45

Perhaps you find it interesting that in the 40 flips of 100 virtual coins, the result 50H, 50T occurred only once? Notice that in the 40 flips of 100 coins, the outcome with the greatest number of H was 63H (with 37T) and the outcome with the least number of H was 40H (with 60T). You might study the two lists of 20 outcomes and wonder if the virtual coin flipper is producing the types of results that would be produced using a true coin.

You might conjecture that it is very unlikely that flipping 100 virtual coins with this particular virtual-coin flipper will produce more that 65 H or fewer than 35 H. This might lead you to do quite a bit more experimentation with this particular piece of software. Many students find it is fun to do such explorations and to share the results with others.

This type of experiment has been carefully explored by math researchers. Mathematicians have developed formulas that can be used to determine the likelihood of various types of outcomes in coin flipping. See http://cnx.org/content/m11020.4/latest/#calc.

Math Maturity Food for Thought. Research suggests that students need to be well along toward achieving formal operations on a Piagetian cognitive development scale to "understand" the mathematics of probability. Students at a much lower cognitive development level can gain some initial understanding of and "feel" for probability and can memorize or figure out some of the simple results in probability. For example, a child playing a board game that uses 2D6 to make moves will gradually realize that a total of 7 occurs much more often that a total of 12.

Here is a question to think about. What math should be directly taught, what math should be taught by a discovery-based approach, and what math should students be encouraged to find and learn on their own? For example, suppose that you are playing a game with a young child, and the game involves moving a playing piece around the board based on rolling 2D6. This learning environment allows you to be a learning facilitator that uses one or more of the three ideas in the question given above. Think about how you decide which to use, and when to use it.

Here are two virtual dice rollers available for free use on the Web.

- The JavaScript Source (n.d.). Games: Dice Roller. Retrieved 6/28/2010 from http://javascript.internet.com/games/dice-roller.html. Choose the number of faces (3, 4, 5, 6, 8, 10, 12, 20, 30, or 100) and the number of dice to roll (1 to 10). The computer produces the sum of the roll.

- Probability (n.d.). Introduction to Probability Models. Retrieved 4/29/2010 from http://www.math.csusb.edu/faculty/stanton/m262/intro_prob_models/intro_prob_models.html Simulates rolling *n*D6 and draws a histogram of the outcome.

Your authors enjoy examining the results from using computerized dice rollers. The types of results we observe lead to observations and questions much like those given above for computerized coin flippers.

Final Remarks

This chapter provides a brief introduction to dice rolling and coin flipping. Dice and coins are examples of math manipulatives. They have many possible uses in helping to increase a student's level of math understanding and math maturity.

True dice and true coins are defined mathematically. A D6 is true if the chances of (the probability of) each of the six possible outcomes from rolling the die are equal—each being 1/6. Similarly, a coin (a D2) is true if the probability of each of the two outcomes from flipping the coin are equal—each being 1/2.

The probabilities of various outcomes are more complex (not so intuitively obvious) when one is rolling 2 or more dice or flipping 2 or more coins. Such problems are studied in probability courses. However, they can also be studied at an experimental level by doing repeated experiments and analyzing the results. Such experiments can be done using real or virtual (computerized) dice and coins.

Activities and Possible Homework Assignments

1. **(For use with students.)** This game is called Roll for Twin Doubles. Each student is provided with 2D6. In each round of this game, each student rolls 2D6, records the outcome of each individual die, and notes whether a doubles occurred. The game ends when a player rolls twin doubles (a particular doubles has been rolled twice) or after a prescribed number of rolls such as 30. If a player rolls twin doubles, the other players are allowed to complete their rolls for that turn, so the game can end in a tie. The minimum length of a game is two rolls, and it is quite possible that the game ends with no twin doubles winner. In that case the winner is determined by who has the most doubles. Figure 9.9 shows a sample record-keeping sheet and part of a game that ended after 25 rolls.

Round Number	One Die	The Other Die	Doubles or Twin Doubles?
1	3	4	no
2	6	6	Double 6
3	1	5	no
4	4	6	no
5	2	2	Double 2
6	2	1	no
7	5	3	no
8	6	1	no
…			
22	3	3	Double 3
23	3	2	no
24	1	3	no
25	6	6	Twin Double 6

Figure 9.9. Recording sheet showing a sample Roll for Twin Doubles game.

A shorter version of the game, Roll for Two Doubles, uses the rule that the game ends when a player has rolled two doubles (that need not be the same). Of course, other players are allowed to complete their rolls in this final turn, so the game can end in a tie.

One purpose of these games is to give students practice in careful record keeping. Another is to help build intuition about how often a doubles occurs and how often a double doubles occurs in a long sequence of rolling 2D6.

Either game can be debriefed in a whole class discussion. Here are some sample questions for the Twin Doubles game.

- Which seems to happen more often: Getting a doubles or not getting a doubles?
- Did anybody get a twin doubles? If yes, what twin doubles did you get, and how many turns did it take?
- Did anybody get all 6 possible doubles, but no twin doubles? If so, how many turns did it take before you had all 6 doubles?
- Did anyone get no doubles?
- Who did not get twin doubles?
- What was fun about this game, and what was not fun about this game?
- What are some things a person can learn by playing this game?

2. **(For use with students.)** This game is called Flip for Triples. Provide each student with three real or "fake" coins. Agree on what constitutes a head (H) and what constitutes a tail (T) on a coin. Play the same type of game as described above. However, in this case the goal is to get two sequences of 3H or two sequences of 3T. In the long version of the game, the game ends when

a player has flipped HHH twice or TTT twice. In the short version of the game, the game ends when a player has flipped any two triples (two HHH or two TTT or one HHH and one TTT).

A much longer variation of this game is to use more coins. With four coins, the goal is to flip for twin quadruples.

Notice that the games given in 1 and 2 do not involve any skill on the part of the payers. These are not "strategy" games. These are games in which students gain skill in recording their moves and increasing their "intuitive feel" for possible outcomes.

3. With practice, a student can develop considerable speed and accuracy in reading the outcome from rolling 2D6. Here is a game for pairs of students. One is the timekeeper (a countdown "kitchen" timer is useful) and also counts the number of rolls the other player processes in the allocated time. The roller rolls 2D6, says the total, and continues to do this over and over again until the allocated time (such as a minute) ends. A variation is to have a third player who counts the number of incorrect answers. Remember, one learning goal is automaticity with both speed and accuracy. Another learning goal is for a student to recognize that both speed and accuracy improve through extended amounts of practice.
4. **(A possible homework assignment in a course.)** Select a dice rolling or coin flipping game or activity mentioned in this chapter. Analyze it from the point of view of the math and writing level of knowledge and skill needed to play game, record the moves, and keep score if the game includes a score-keeping mechanism. Continue your analysis from the points of view of how playing the game or doing the activity might contribute to a student learning math or gaining in math maturity.
5. **(A possible homework assignment in a course.)** Make up two dice and/or coin games/activities. Carefully specify the rules, record keeping, and scoring. Then analyze each in terms of possible contributions to student math learning and student increase in math maturity.

Chapter 10: Place Value Games

Introduction

Of course you know about place values of whole numbers. A one-digit number has only a **ones place**, a two-digit number has a **tens place** and a **ones place**, a three digit number has a **hundreds place**, a **tens place**, and a **ones place**. More-digit numbers have more places.

To review various place-value topics, see http://www.aaamath.com/plc.html. This Web site presents interactive place value tutorials using several methods of presentation. It begins with two-digit numbers, then three-digit numbers, gently moves on to more-digit whole numbers, and culminates in decimal place values (tenths, hundredths, et cetera, et cetera).

In this chapter, we describe two-digit place-value games and three-digit place-value games. You can easily extend these games to four digits, five digits, and more digits. These games can be played with a variety of dice, such as D6 (6-faced dice), D8 (8-faced dice) and D10 (10-faced dice numbered 0 to 9).

This chapter includes an emphasis on learning, analyzing, and using strategies that help one to make good decisions. Games provide environments in which students can learn strategies on their own and can learn strategies from other people.

Two-Digit Place-Value Games Using 1D6

In introducing children and teachers to place-value games, we like to begin with easy games and work our way up to more difficult games. We are now ready to present a two-digit place-value game that uses one six-faced die (1D6). Using this quite simple game, we will lay foundations for more complex and challenging place-value games, and the development of strategies for playing place-value games.



1D6

In these games, a 1D6 is rolled twice in a process of creating a two-digit number. One of the die rolls is used to make the tens digit and the other to make the ones digit.

- The number 66 has 6 in the tens place and 6 in the ones place. It is the largest two-digit number you can roll using 1D6 for each roll.
- The number 11 has 1 in the tens place and 1 in the ones place. It is the smallest two-digit number you can roll using 1D6 for each roll.
- The number 25 has 2 in the tens place and 5 in the ones place. The number 52 has 5 in the tens place and 2 in the ones place.

In the next two sections we describe and explore two different 2-player place-value games that are played using 1D6. The games are named **Try for High** and **Go for Low**. In each of these games a player rolls 1D6 and makes a decision as to whether the outcome should be used as the tens place or the ones place. The player than rolls 1D6 again and uses its outcome as the other digit in a 2-digit number. The player needs to make a decision after his or her first roll. This is not a simple decision, because the player does not yet know the result of his or her second die roll.

Two-Digit Try for High Game

In the Try for High two-player game, the objective is to try to get a higher (greater) two-digit number than one's opponent.

Maia and Joel are teachers who are cruising the Oregon coast during summer break. They have both taught math and science at various levels, and are now focusing on games as tools and toys for learning and teaching. During their slow tour down Oregon Coast Highway 101 from Astoria in the north to Harbor in the south, they will hike, camp, explore, play games, analyze game mathematically, write instructional stuff, et cetera, et cetera. A glorious, wonderful summer! Along the way, they share what they learn.

Maia and Joel play the Try for High Place Value game. In a two-player game, each player tries to get a larger 2-digit number than her or his opponent

Suppose that the players can see each other's moves as they play. In the example below, Maia rolls 1D6 twice to get her 2-digit number, and then Joel rolls 1D6 twice to get his 2-digit number. Joel eagerly awaits seeing Maia's before he rolls for his number. Maia is OK with Joel's smug anticipation.

Maia, game 1:	***tens***	***ones***
1st roll. Maia rolled 1D6 and got 5. She placed it in the tens place. Do you agree with her choice of place value?	5	
2nd roll. She rolled 1D6, got 3, and placed it in the remaining place, the ones place. Her try for high 2-digit number is 53.	5	3

Joel, game 1:	***tens***	***ones***
1st roll. Joel rolled 1D6 and got 4. He placed it in the ones place. Do you agree with his choice of place value?		4
2nd roll. He rolled 1D6 and, alas, got 2. He placed it in the remaining place, the tens place. His try for high 2-digit number is 24.	2	4

Maia won the first game. It may be that Joel has some advantage in playing second. When he saw that his first roll was a 4, he decided he would use it as the ones digit. If he uses it as the tens digit, he will lose the game for sure. By using the 4 as the ones digit, he still has a chance to win. If his second roll is a 5 or a 6, he will win the game. This analysis illustrates a simple but useful strategy for the second player. A strategy is a systematic plan of action. All of us routinely make use of strategies we develop and/or that we have learned from others.

This analysis also suggests that players should take turns going first. There is an advantage to being the second player in this game. However, Maia and Joel decided that Maia would go first for one more game just to see what happens.

Maia, game 2:	***tens***	***ones***
1st roll. Maia rolled 1D6 and got 4. She placed it in the tens place. Do you agree with her choice of place value?	4	
2nd roll. She then rolled 1D6, got 6, and placed it in the remaining place, the ones place. Her try for high two-digit number is 46.	4	6

Joel, game 2: 1st roll. Joel rolled 1D6 and got 4. He thinks, "Grrr! If I put it in the tens place, I will have to roll 6 just to tie. Most unlikely." He placed his 4 in the ones place. Do you agree with his choice?	***tens***	***ones*** 4
2nd roll. He rolled 1D6 and got 5. With a huge smile, he placed it in the remaining place, the tens place. His try for high two-digit number is 54.	5	4

Joel won the second game. He followed the strategy of using the first outcome as the ones digit if it is smaller than or equal to the tens digit of his opponent. His lucky second roll produced a high enough outcome to win the game.

In retrospect, it is easy to claim that Maia's first choice was a poor choice. If she had used her first roll is the ones digit, she would have won the game. It is easy to second-guess people's choices when they make decisions!

In the first two games, the first player completed her or his turn (two rolls of 1D6) before the second player began. The second player has a big advantage! Maia and Joel decided that it would be better if the players each roll 1D6 for the first digit (tens or ones), and then each roll 1D6 again for the second digit, like this:

1st player rolls 1D6 and places the outcome in her or his tens place or ones place.

2nd player rolls 1D6 and places the outcome in his or her tens place or ones place.

1st player rolls 1D6 and places the outcome in her or his remaining place, the place not chosen in the first roll.

2nd player rolls 1D6 and places the outcome in his or her remaining place, the place not chosen in the first roll.

This method is used in Try for High Game 3.

Try for High Game 3	Joel		Maia	
	tens	ones	tens	ones
Joel's first roll: He rolled 1D6 and got 4. Always optimistic that he can beat the odds, he placed it in the ones place. Do you agree with his choice? What is the probability that he can roll 5 or 6 on his next roll? What is the probability that he can roll 4, 5, or 6 on his next roll?		4		
Maia's first roll: She rolled 1D6 and got 5. She thinks, "If I place it in the tens place, Joel will have to roll 5 or 6 to tie or win." She placed her 5 in the tens place, an easy decision.			5	
Joel's second roll: Your turn. Pretend you are Joel. Roll 1D6 for Joel and place the outcome in the remaining place, the tens place.		4		
Maia's second roll: Your turn. Pretend you are Maia. Roll 1D6 for Maia and place the			5	

outcome in the remaining place, the ones place.				

Who won? Only you know. If you roll a 6 for Joel, he will win no matter what you roll for Maia. If you roll 1, 2, 3, or 4 for Joel, then Maia will win for sure. If you roll a 5 for Joel, then what must you roll for Maia so that she will win? We encourage you and your students to play games such as Try for High Game 3 in which they take turns for their first rolls, and then take turns again for their second rolls.

Math Maturity Food for Thought. In Game 3 above, the second player sees the first player's die roll and how it was placed, before making a decision about how to use his or her first roll. Does this give the second player an advantage? If yes, is it a small advantage or a big advantage? Suppose both players roll 1D6 at the same time with a screen between them so that they cannot see each other's outcomes. Each player places his or her outcome in the tens place or the ones place, then the screen is removed so that they both see the other's die roll and where it was placed. Does either player have an advantage? Is this a fair approach with neither player having an advantage? Discuss various ways to play the game so that 1) one player has an advantage or 2) neither player has an advantage.

Two-Digit Go for Low Game

The Go for Low game follows the same rules as the Try for High game, except the objective is to get a lower (smaller) two-digit number than one's opponent. As in the Try for High games, the second player has the advantage of seeing the first player's moves. Maia and Joel decide to alternate die rolls as they did in their Try for High Game 3. Here are two sample games.

Go for Low Game 1	Maia		Joel	
	tens	ones	tens	ones
Maia's first roll: She rolled 1D6 and got 5. She placed it in the ones place. Do you agree with her choice? What is the probability that she can roll less than 5 on her next roll?		5		
Joel's first roll: He rolled 1D6 and got 4. He is confident that he can roll less than 4 on his next roll, so he places his 4 in the ones place. Do you agree with his decision?				4
Maia's second roll: She rolled 1D6 and got 3. She placed the outcome in the remaining place, the tens place. Her Go for Low 2-digit number is 35.	3	5		
Joel's second roll: Your turn. Pretend you are Joel. Roll 1D6 for Joel and place the outcome in his tens place.				4

Who won? Only you know. If you rolled 1, 2, or 3 for Joel, he wins. If you rolled 4, 5, or 6, he loses. What is the probability that Joel will win? What is the probability that he will lose?

Go for Low Game 2	Joel		Maia	
	tens	ones	tens	ones
Joel's first roll: He rolled 1D6 and got 2. He placed it in the tens place. Do you agree with his choice?	2			
Maia's first roll: She rolled 1D6 and got 3. She knows that she needs a 1 or a 2 in the tens place to beat or tie Joel, so she puts her 3 in the ones place. Do you agree with her decision?				3
Joel's second roll: He rolled 1D6 and got 4. He placed it in the remaining place, the ones place.	2	4		3
Maia's second roll: Your turn. Roll 1D6 for Maia and place the outcome in her tens place.	2	4		3

Who won? Only you know. If you rolled 1 or 2 for Maia, she wins. If you rolled 3, 4, 5, or 6, she loses. What is the probability that Maia will win? What is the probability that she will lose? We encourage you and your students to play games such as the Go for Low games above in which they take turns for their first rolls, and then take turns again for their second rolls.

There is an advantage to being the second player in this game, so Maia and Joel suggest that players take turns going first, and also take turns rolling for each digit. In the next section, we suggest scenarios for classroom games in which no player has an advantage.

Math Maturity Food for Thought. The Try for High and Go for Low games described above are quite easy to learn how to play. Initially, a beginner might make random choices, or adopt a poor strategy. Over time, the player's mind may detect a pattern of not doing as well as she or he would like and deduce that the strategy is not a good one.

Eventually a player may figure out that if the first roll in Try for High is 1, 2, or 3, it might be good to use it as the ones digit. This is an example of discovery-based learning based on mentally analyzing one's choices and the outcome of a choice turning out to be a good or a poor choice. Wow! This gives some insight into the challenge of discovery-based learning.

Contrast this with an adult telling the student: "If you are the first player in Try for High and your first roll is 1, 2, or 3, use this as the ones digit." The adult then goes on to give a number of other strategies, all with little or no explanation of reasons behind the strategies. In essence, the student is told, "Follow the rules I give you and you will win more often."

Think about how this analysis applies to the teaching and learning of math. In some sense, this situation lies at the very heart of the Math Education Wars. (See http://iae-pedia.org/Math_Education_Wars.)

Classroom Scenarios

Maia and Joel are teachers. They suggest classroom scenarios in which players do not see the moves of other players. In these scenarios, the use of systematic strategies becomes important and students can be encouraged to create strategies that will enhance their chances of scoring larger numbers in the Try for High game and smaller numbers in the Go for Low game.

Scenario #1. The players are teams of two or three students. Each team can see only its own play sheet, and cannot see what any other team is doing. The teacher or a designated student rolls the die. The die roller might be called a **Dice Master** or **Game Master**. This could be a coveted position that students might strive to attain.

> 1st die roll. Each team decides where to place the outcome of the first die roll, tens place or ones place, and does so.
>
> 2nd die roll. Each team places the outcome of the second die roll in the place not chosen in the 1st roll.

If the game is Try for High, the winners are teams with the greatest 2-digit number. There may be more than one team with the largest 2-digit number.

If the game is Go for Low, the winners are teams with the smallest 2-digit number. There may be more than one team with the smallest 2-digit number.

Aha! That suggests a variation that we like. After the first roll, let each team choose to play Try for High or Go for Low. That is, instead the game being Try for High with multiple teams competing, let each team look at the first roll and decide whether it is playing Try for High or Go for Low. If at least one team decides on Try for High and at least one team decides on Go for Low, one or more teams will win the Try for High game and one or more will win the Go for Low game. Can you think of a way to determine a "grand winner" by specifying a way of comparing the highness of a Try for High winner versus the lowness of a Try for Low winner?

Scenario #2. The class is divided into groups of three or more students. In each group, one student is the Dice Master and rolls the die. The other students in a group are players. Before beginning the game, the Dice Master tells the players if the game is Try for High or Go for Low, of if they can choose either game after the first roll.

Each player can see only her or his own play sheet, and cannot see the other players' play sheets. After the Dice Master's first roll, each player places the outcome in the tens place or the ones place on his or her play sheet. All players in a group use the same data and the game is a within-group competition.

The Dice Master rolls 1D6 again and the players complete their 2-digit numbers by placing the outcome in the place not chosen for the first roll.

Scenario #3. In this two-player game, players can see their opponents play sheet. They play, say, 10 games, alternating as first player—each player is first player for five games and second player for five games. In Try for High, the winner is the player with the greatest total score in 10 games. In Go for Low, the winner is the player with the lowest total score for 10 games. Each player must keep his or her eye on the goal, the better total score in 10 games.

Math Maturity Food for Thought. In Scenario #3, the players take turns going first, and the goal is to achieve the better score for the total sequence of 10 games. Analyze this approach to figure out if there is an advantage to being the first player to go first or the second player to go first. Select some other games (including sports games) from the point of view of whether there is an advantage or a disadvantage to going first.

Strategies and Decision Making

Throughout your everyday life you make decisions and then act on these decisions. Businesses, governments, and other organizations also make decisions. Generally speaking, people and organizations make decisions that they feel, think, or believe will lead to outcomes that are favorable to themselves or to some cause they support. They have developed and/or learned strategies they find helpful in making good decisions.

As a child you likely learned the strategy, "Look both ways before starting to cross a street." This strategy was taught to you before you had learned about the perils of vehicular traffic. There is a good chance that you now use this strategy even when crossing a one-way street or crossing at a traffic light.

Math Maturity Food for Thought. Think about your informal and formal education. To what extent are your insights into decision making drawn from the "school of hard knocks?" To what extent have you received careful informal and formal education on how to make good decisions and then to take responsibility for your decision?

Think about the somewhat routine and mundane decisions you make in a typical day. What strategies do you use in helping to make these decisions? Probably you don't use coin flipping or dice rolling to make these decisions, but probably you don't spend a lot of time thinking about them. They have become routinized.

However, from time to time you are faced by a major decision. Think about the processes you use in making large, important decisions. This thinking will lay groundwork for the remainder of this section.

Solving a multistep math problem requires carrying out a sequence of actions. At each step, you need to make a decision about what to do next. That is, problem solving in math or any other area requires a sequence of carefully reasoned decisions. Many games—such as the ones discussed in this book—provide an environment in which a person can learn more about decision-making.

The decisions we make affect and shape our futures. Thus, typically the process of making a decision involves giving some thought about possible alternate decisions and possible outcomes from the various possible choices. Thus, we are all futurists. We all spend time trying to predict and shape the future.

Suppose you need to make an important decision. Part of your strategy may be to understand the nature of the decision that you are going to make and to think about possible outcomes. Another part of your strategy may be to talk to people who have been faced by somewhat similar situations. A third part may be to make use of the Web to look up information related to the decision you need to make. In making your final decision you use information from yourself, other people you are in contact with, and information resources such as the Web.

Helping Students Learn to Develop Strategies

Games provide an environment in which students can create and try strategies, and can also learn strategies from others. As a teacher, you must routinely make decisions about helping your students learn to develop strategies on their own versus having your students learn strategies that have been developed by you and others. You realize, of course, that one size does not fit all, so you try to figure out the individual needs of your students. However, keep in mind that a major goal is to help your students learn about creating and testing strategies, and in using strategic thinking.

What follows are some examples of strategies that Maia and Joel develop. In the place-value game environment, the goal is to learn about creating and testing strategies.

As you encourage your students to learn about creating, using, and testing strategies, you will need to have some mechanism for your students to communicate their thinking and results to you and others. For example, this might be done orally, via a written paper or email, or by journal writing. And, of course, you can observe the decisions your students are making and perhaps detect when they appear to be using a questionable strategy.

Maia's Two-Digit Try for High Strategy

Maia has developed a strategy to use when she goes first in playing 1D6 Try for High or when she cannot see the moves of another player. Maia describes her strategy like this:

> 1st roll. If the result is 4, 5, or 6, I place it in the tens place. If the result is 1, 2, or 3, I place it in the ones place.
>
> 2nd roll. I place the result in the remaining place value, the place not chosen in the first roll.

It does not require deep thinking to make use of this strategy, and it is easy to teach this strategy to others. Indeed, a computer can be programmed to follow this strategy in playing Try for High.

Is it a good strategy? Is it the best possible strategy? How can Maia test her strategy? Aha! She knows two good ways. One way: play lots of solitaire games, keep good records, and calculate statistics. Another way: calculate the **mathematical expected value** of her strategy.

These types of strategy questions help to define an important branch of mathematics named Game Theory that was developed in the 1940s. See http://en.wikipedia.org/wiki/Game_theory. Quoting from the Wikipedia:

> Game theory is a branch of applied mathematics that is used in the social sciences, most notably in economics, as well as in biology (most notably evolutionary biology and ecology), engineering, political science, international relations, computer science, and philosophy. **Game theory attempts to mathematically capture behavior in strategic situations, in which an individual's success in making choices depends on the choices of others.** [Bold added for emphasis.]

Maia has entered into one of the most important areas of applied mathematics. She will explore this field using ideas that she figures out for herself, learns by collaborating with Joel, and gleans from her old math books and the Internet.

Maia's initial thoughts are to try to estimate the average two-digit number that her strategy produces if she plays a bunch of games. She decided to gather some experimental data, played 10 games using her strategy, and recorded her scores.

43, 31, 44, 32, 62, 25, 61, 51, 41, 52

Maia's total score for 10 games was 442. Her mean score per game was 442/10 = 44.2. She played another 10 games and got a total score of 469. Her mean score for that set of 10 games was 469/10 = 46.9.

"Interesting," mused Maia. "If I play lots of games using my strategy, what might be my average two-digit number?" So she did. Maia played 100 games. The sum of her 100 two-digit numbers was 4513. The mean of her 100 two-digit numbers was 4513/100 = 45.13.

Maia happily spent time relearning how to calculate a precise mathematical calculation of the **expected value** of a two-digit number using her try for high strategy. The mathematical expected value is 45.25. If she played thousands of games using her strategy, she would 'expect' a mean score near 45.25, maybe a tad more, perhaps a bit less, but in the neighborhood of 45.25.

We have not provided details on Maia's math in computing this expected value. We encourage you to search the Internet for *expected value*. You can find Maia's mathematical analysis and 100-game play test statistics in our online supplements at http://iae-pedia.org/Supplementary_Materials_for_2011_Book_by_Moursund_and_Albrecht.

Maia realizes that having an estimate of her expected value does not give any information about how often she should expect to win, tie, or lose. If she plays against another player and they cannot see each other's moves, then it is her strategy versus the strategy of the other player. She will explore other strategies and encourages you and your students to explore strategies.

Joel's Two-Digit Try for High Strategy

Maia and Joel are a team—they would rather collaborate than compete. Maia shared her strategy with Joel and encouraged him to develop a different strategy that they could test.

Joel's first thought is that Maia's strategy is a tad conservative. He thinks, "Perhaps it would be better to only use the first roll in the tens place when it is a 5 or a 6. After all, there is a reasonable chance of getting a 5 or a 6 in the second roll. With such a strategy, I stand a greater chance of getting a 2-digit outcome that begins with a 5 or a 6, and perhaps that will lead me to winning more often."

Here is a description of Joel's strategy:

1st roll: If the result is 5 or 6, I place it in the tens place. If the result is 1, 2, 3, or 4, I place it in the ones place.
2nd roll: I place the result in the remaining place value, the place not chosen in the first roll.

As did Maia, Joel gathers experimental data. Joel played 10 games and recorded his scores.

13, 66, 62, 51, 42, 34, 63, 33, 14, 54

Joel's total score for 10 games was 432. His mean score per game was 432/10 = 43.2. He was surprised that his mean score was less than produced by Maia's strategy, so he played another 10 games and got a total score of 451. His mean score for that set of 10 games was 451/10 = 45.1.

Maia challenged Joel to test his strategy by playing 100 games. After a bit of grumbling, he did it. His total score for 100 games was 4,401. His mean score per game was 4,401/100 = 44.01. Interesting! Based on a play of 100 games, the experimental expected value from Maia's strategy is slightly higher that the experimental expected value from Joel's strategy.

Maia's mean score for 100 turns: 45.13

Joel's mean score for 100 turns: 44.01

Math Maturity Food for Thought. The precise mathematical expected value from Joel's strategy is 44.50 per turn, slightly less than the mathematical expectation of Maia's strategy (45.25). That suggests that Maia's strategy is slightly better than Joel's strategy.

However, it does not prove that Maia will win more often than Joel if Maia follows her strategy and Joel follows his strategy. See if you can figure out why this is the case. Can you provide examples from season-long team sports in which a team with the highest average score per game does not compile the best win/lose record?

This is a good example of when one's intuition can be wrong. Think about sporting events such as football, basketball, hockey, and baseball. While a team's average score is a useful measure of its performance, a team may well "win big" (win by a large margin) when winning, and lose by a smaller margin when losing. A team with a smaller average score may win more often!

Joel and Maia could gather more experimental evidence on this situation by playing a large number of games, each using their own strategy. Alternatively, a computer program could be used to play a large number of games and thus to make an experimental estimate of the probability.

Figure 10.1 summarizes the mathematical expected values and 100-game play-test results for Maia's and Joel's 2-digit Try for High strategies. The table also compares the expected values and play-test results by showing the ratios of play-test results to expected values, and the percent differences of play-test rest results from expected values.

	Maia's strategy	Joel's strategy
Mathematical expected value	45.25	44.50
Play test: 100-game total score	4513	4401
Play test: 100-game mean score per game	4513 / 100 = 45.13	44.01

Figure 10.1. Expected values and play test results for Try for High strategies

In a classroom game environment in which a player cannot see another player's play sheet, strategies become very important. If Maia and Joel played hundreds of games using their strategies, Maia would likely win a few more games than Joel.

Go for Low Strategies

Maia and Joel decided to develop Go for Low strategies that are opposites of their Try for High strategies.

Maia's Go for Low strategy:
1st roll. If the result is 1, 2, or 3, I place it in the tens place. If the result is 4, 5, or 6, I place it in the ones place.
2nd roll. I place the result in the remaining place value, the place not chosen in the first roll.

Joel's Go for Low strategy:
1st roll: If the result is 1 or 2, I place it in the tens place. If the result is 3, 4, 5, or 6, I place it in the ones place.
2nd roll: I place the result in the remaining place value, the place not chosen in the first roll.

They used their Go for Low strategies in a few games in which they could not see each others play sheets and did not know each others moves. Instead of playing more games, Maia and Joel tested each strategy by playing 100 turns of each strategy. They also calculated the mathematical expected values for their strategies and compared the expected values and play-test values. Their results are shown in Figure 10.2.

	Maia's strategy	Joel's strategy
Mathematical expected value	31.75	32.50
Play test: 100-game total score	3285	3066
Play test: 100-game mean score per game	32.85	30.66

Figure 10.2 Expected values and play test results for Maia's and Joel's Go for Low strategies

Math Maturity Food for Thought. The above section of decision-making and strategies may well be the most important part of this book. Simple games can be used to help students learn about developing, using, and testing strategies. The teaching goal is to make these ideas explicit. The underlying area of mathematics—called Game Theory— has proven useful in many different disciplines.

You have probably heard of the Math Education Wars. See http://iae-pedia.org/Math_Education_Wars. A major conflict between the various parties involved in these wars is illustrated in the discussion about explicitly teaching strategies versus involving students in discovering, using, and testing strategies.

Analyze your current math knowledge and skills and your math teaching in terms of a strategies point of view. Try to briefly summarize your overall strategy in math teaching and reconcile it with the types of arguments being given in the Math Education Wars.

Three-Digit Place-Value Games

This section explores 3-digit place value games played using 1D6. Of course, the same games can be played with 1D4, 1D8, or 1D10. It is reasonably easy to transfer one's learning from the 1D6 environment to use of other dice.

A three-digit number has a **hundreds place**, a **tens place**, and a **ones place**.

	hundreds	***tens***	***ones***
The number 666 has 6 in the hundreds place, 6 in the tens place, and 6 in the ones place. 666 is the largest three-digit number you can roll using 1D6 for each roll.	6	6	6
The number 111 has 1 in the hundreds place, 1 in the tens place, and 1 in the ones place. 111 is the smallest three-digit number you can roll using 1D6 for each roll.	1	1	1
The number 235 has 2 in the hundreds place, 3 in the tens place, and 5 in the ones place.	2	3	5

Three-digit games are similar to the previously described two-digit games. You roll 1D6. The result is a number 1 to 6. Place the result in the hundreds place, the tens place, or the ones place to begin forming a three-digit number. Roll 1D6 again and place the result in one of the two place-value positions not selected in the first roll. Roll 1D6 a third time and place the result in the one-and-only remaining place value.

Notice that a player needs to make two different decisions in playing the game: how should the first roll be used and how should the second roll be used? The need to make two decisions increases the complexity of the game and provides an increased challenge in developing and testing possible strategies.

Try for High, Three-Digit Game

Objective: Try for the highest (greatest) three-digit number. Using 1D6 for each roll, the greatest possible number is 666. Maia and Joel will play sample games. Instead of **hundreds**, **tens**, and **ones** as the place value headings, we use **100**, **10**, and **1**, respectively.

Try for High Game 1. Maia is the 1st player	Maia			Joel		
	100	**10**	**1**	**100**	**10**	**1**
1st player. Maia rolled 1D6 and got a 5. She placed it in the hundreds place. Do you agree with her choice?	5					
2nd player. Joel rolled 1D6 and got 4. If he places it in the hundreds place, he is sure to lose. He placed it in the ones place. Do you agree with his choice?						4
1st player. Maia rolled 1D6 and got 2. She placed it in the ones place. Do you agree with her choice?	5		2			
2nd player. Joel rolled 1D6, got 3, and placed it in the ones					3	4

place. Do you agree with his choice?						
1st player. Maia rolled 1D6 and got 6. She placed it in the remaining place, the tens place. Her three-digit try for high number is 562.	5	6	2			
2nd player. Joel rolled 1D6 and got 2. He reluctantly placed in the remaining place, the hundreds place. His three-digit try for high number is 234.				2	3	4

Maia won the first game. Rolling a 5 in the first roll was a great start! Joel rolled a 4 in his first turn. If he placed it in the hundreds place, he would definitely lose, so he put it in the tens place. Besides, Joel is always confident that he can beat the odds. Having lost the first game, Joel wants a rematch.

Try for High Game 2. Joel is the 1st player	Joel			Maia		
	100	**10**	**1**	**100**	**10**	**1**
1st player. Joel rolled 1D6 and got 4. He placed it in the 10s place. Do you agree with his choice?			4			
2nd player. Maia rolled 1D6 and got 4. She placed it in the hundreds place. Do you agree with her choice of place value?				4		
1st player. Joel rolled 1D6, got 3, and placed it in the ones place. Do you agree with his choice?		3	4			
2nd player. Maia rolled 1D6 and got 3. She placed it in the ones place. Do you agree with her choice?				4		3
1st player. Joel rolled 1D6, got 6, and triumphantly placed it in the remaining place, the hundreds place. His three-digit try for high number is 643.	6	4	3			
2nd player. Maia rolled 1D6 and got 6. She placed it in the remaining place, the tens place. Her three-digit try for high number is 463.				4	6	3

Joel won the second game. As you might see, his strategy places emphasis on trying for a high number in the hundreds place. In this second game this strategy paid off – he rolled a 6 in the third roll. What is the probability of rolling a 6?

Go for Low, Three Digit Game

Objective: Go for the lowest three-digit number. Using 1D6 for each roll, the lowest possible number is 111. Maia and Joel demonstrate with two games.

Go for Low Game 1. Joel is the 1st player	Joel			Maia		
	100	**10**	**1**	**100**	**10**	**1**

1st player. Joel rolled 1D6 and got 2. He placed it in the hundreds place. Do you agree with his choice?	2					
2nd player. Maia rolled 1D6 and got 5. She placed it in the ones place. Do you agree with her choice?						5
1st player. Joel rolled 1D6 and got 3. He placed it in the tens place. Do you agree with his choice?	2	3				
2nd player. Maia rolled 1D6 and got 4. She placed it in the tens place. Do you agree with her choice?					4	5
1st player. Joel rolled 1D6 and got 5. He placed it in the remaining place, the ones place. His three-digit go for low number is 235.	2	3	5			
2nd player. Maia rolled 1D6 and got 2. She placed in the remaining place, the hundreds place. Her three-digit go for low number is 245.				2	4	5

Joel won the first game by a tad. He was lucky to roll a 2 on his first roll. What is the probability of rolling 1 or 2 on the first roll? Now Maia has an uphill battle – she must roll 1 or 2 in the next two rolls to win or tie. She rolled a 2 in her third roll but, alas, Joel had a lower tens digit and Maia lost.

Go for Low Game 2. Maia is the 1st player	Joel			Maia		
	100	**10**	**1**	**100**	**10**	**1**
1st player. Maia rolled 1D6 and got 4. She placed it in the ones place. Do you agree with her choice?			4			
2nd player. Joel rolled 1D6 and got 4. He placed it in the ones place. As always, he is confident that he can roll 1 or 2 in the next two rolls. Do you agree with his choice?						4
1st player. Maia rolled 1D6 and got 3. She placed it in the hundreds place. Do you agree with her choice?	3		4			
2nd player. Joel rolled 1D6 and got 4. Still optimistic about his next roll, he placed it in the tens place. Do you agree with his choice?					4	4
1st player. Maia rolled 1D6 and got 4. She placed it in the remaining place, the tens place. Her three-digit go for low number is 344.	3	4	4			
2nd player. Joel rolled 1D6 and got 3. With a sigh of relief, he placed in the remaining place, the hundreds place. His three-digit go for low number is 344. It's a tie!				3	4	4

Maia and Joel are far more interested in developing and testing strategies than just playing more games. You will see their strategies in the next section.

In Try for High, a player tends to think in terms of numbers that are larger than other numbers. In Go for Low, a player tends to think in terms of numbers that are smaller than other numbers. How does one tell whether one 3-digit integer is larger (or smaller, or the same) as another? Both the Try for High and the Go for Low give players practice in making mental comparisons of 3-digit natural numbers.

Math Maturity Food for Thought. We illustrated both the two-digit value-value games and three-digit place-value games using 1D6. Think about the following two questions:

1. How much of a challenge will your students face as they move from playing 1D6 Try for High to 1D6 Go for Low? How might you use such a pair of closely related games to talk about and illustrate transfer of learning? Think about other examples in which you expect your students to readily do a transfer of learning.
2. How much of a challenge is it for students to move from playing the 2-digit version to the 3-digit version of Try for High or Go for Low? Analyze the situation from a transfer of learning point of view.
3. Consider the idea of having and/or developing strategies to help people make good decisions in other decision-making situations. What do your students know about this idea? What do you do to help them increase their insights into learning and using strategies, and that some strategies are better than others?

Strategies in Three-Digit Place-Value Games

Maia and Joel gleefully developed strategies for three-digit games that are similar to their strategies for two-digit games. They explain their 3-digit strategies.

Maia's Try for High strategy:

1st roll. If the outcome is 4, 5, or 6, I place it in the hundreds place. If the outcome is 1, 2, or 3, I place it in the ones place.

2nd roll.

If the outcome is 4, 5, or 6 and the hundreds place is available, I place it there. If the hundreds place is already filled, I put 4, 5, or 6 in the tens place.

If the outcome is 1, 2, or 3 and the ones place is available, I place it there. If the ones place is already filled, I place 1, 2, or 3 in the tens place.

3rd roll. I place the outcome in the remaining place, the place not chosen in the first two rolls.

Maia's Go for Low strategy:

1st roll. If the outcome is 1, 2, or 3, I place it in the hundreds place. If the outcome is 4, 5, or 6, I place it in the ones place.

2nd roll.

If the outcome is 1, 2, or 3 and the hundreds place is available, I place it there. If the hundreds place is already filled, I put 1, 2, or 3 in the tens place.

If the outcome is 4, 5, or 6 and the ones place is available, I place it there. If the ones place is already filled, I place 4, 5, or 6 in the tens place.

3rd roll. I place the outcome in the remaining place, the place not chosen in the first two rolls.

Joel's Try for High strategy:

1st roll. If the outcome is 5 or 6, I place it in the hundreds place. If the outcome is 1, 2, 3, or 4, I place it in the ones place.

2nd roll.

If the outcome is 4, 5, or 6 and the hundreds place is available, I place it there. If the hundreds place is already filled, I put 4, 5, or 6 in the tens place.

If the outcome is 1, 2, or 3 and the ones place is available, I place it there. If the ones place is already filled, I place 1, 2, or 3 in the tens place.

3rd roll. I place the outcome in the remaining place, the place not chosen in the first two rolls.

Joel's Go for Low strategy:

1st roll. If the outcome is 1 or 2, I place it in the hundreds place. If the outcome is 3, 4, 5, or 6, I place it in the ones place.

2nd roll.

If the outcome is 1, 2, or 3 and the hundreds place is available, I place it there. If the hundreds place is already filled, I put 1, 2, or 3 in the tens place.

If the outcome is 4, 5, or 6 and the ones place is available, I place it there. If the ones place is already filled, I place 4, 5, or 6 in the tens place.

3rd roll. I place the outcome in the remaining place, the place not chosen in the first two rolls.

Maia and Joel love to create and test strategies. They enjoy having reached a level of math maturity that supports their “playing around” with these types of math problems.

They believe that a play test of 100 games will provide a good experimental estimate of the expected score per game. They rolled 100 turns using each strategy and also applied their math wizardry to developing mathematical expectations for each strategy. Their results are summarized in Tables 10.3 and 10.4.

	Maia's strategy	Joel's strategy
Mathematical expected value	499.875	501.75
Play test: 100-game total score	49,878	50,031
Play test: 100-game mean score per game	498.78	500.31

Figure 10.3 Expected values and play test results for 3-digit Try for High strategies

For both Try for High strategies, the experimental play-test mean values are very close to the mathematical expected values (less than 1%), closer than Maia and Joel expected (pun intended). They would have been happy if play-test values were within a few percent of mathematical expected values, and know that other 100-game play tests might produce mean scores farther from the mathematical expectations. We encourage you and your students to play test this game and report your findings online. Maia and Joel will continue play testing their strategies. At the end of this section, they will suggest strategies for you to test.

	Maia	Joel
Mathematical expected value	277.125	275.25
Play test: 100-game total score	28,285	26,802
Play test: 100-game mean score per game	282.85	268.02

Figure 10.x4 Expected values and play test results for 3-digit Go for Low strategies

Maia's and Joel's play-test mean scores are within 2% to 3% of their mathematical expected values. Maia's play-test score is a little more than her expected value, and Joel's is a little less that his expected value.

Maia's and Joel's mathematical expected values for both Try for High and Go for Low are almost the same. They differ by less than 1%. So, if they play many two-player games, the vagaries of fate (the luck of the die) might favor either one.

Do It Yourself

Are there better strategies? Maia and Joel don't know, but they continue to invent alternate strategies to test. In the meantime, they invite you and your students to invent and test strategies, and suggest the following strategies to help you get started.

M & J's Try for High strategy for you to test:

1st roll. If the outcome is 5 or 6, place it in the hundreds place. If the outcome is 3 or 4, place it in the ones place. If the outcome is 1 or 2, place it in the ones place.

2nd roll. If the outcome is 4, 5, or 6 and the hundreds place is available, place it there. If the hundreds place is already filled, put 4, 5, or 6 in the tens place. If the outcome is 1, 2, or 3 and the ones place is available, place it there. If the ones place is already filled, place 1, 2, or 3 in the tens place.

3rd roll. Place the outcome in the remaining place, the place not chosen in the first two rolls.

M & J's outrageously optimistic Try for High strategy for you to test:

1st roll. If the outcome is 6, place it in the hundreds place. If the outcome is 4 or 5, place it in the tens place. If the outcome is 1, 2, or 3, place it in the ones place.

2nd roll. If the outcome is 5 or 6 and the hundreds place is available, place it there. If the hundreds place is already filled, put 5 or 6 in the tens place. If the outcome is 3 or 4 and the tens place is available, place it there; otherwise, put it in the ones place. If the outcome is 1 or 2, and the ones place is available, put it there. If the ones place is already filled, place 1 or 2 in the tens place.

3rd roll. Place the outcome in the remaining place, the place not chosen in the first two rolls.

M & J's Go for Low strategy for you to test:

1st roll. If the outcome is 1 or 2, place it in the hundreds place. If the outcome is 3 or 4, place it in the ones place. If the outcome is 5 or 6, place it in the ones place.

2nd roll. If the outcome is 1, 2, or 3 and the hundreds place is available, place it there. If the hundreds place is already filled, put 1, 2, or 3 in the tens place. If the outcome is 4, 5, or 6 and the ones place is available, place it there. If the ones place is already filled, place 4, 5, or 6 in the tens place.

3rd roll. Place the outcome in the remaining place, the place not chosen in the first two rolls.

M & J's outrageously optimistic Go for Low strategy for you to test:

1st roll. If the outcome is 1, place it in the hundreds place. If the outcome is 2 or 3, place it in the tens place. If the outcome is 4, 5, or 6, place it in the ones place.

2nd roll. If the outcome is 1 or 2 and the hundreds place is available, place it there. If the hundreds place is already filled, put 1 or 2 in the tens place. If the outcome is 3 or 4 and the tens place is available, place it there; otherwise, put it in the ones place. If the outcome is 5 or 6, and the ones place is available, put it there. If the ones place is already filled, place 5 or 6 in the tens place.

3rd roll. Place the outcome in the remaining place, the place not chosen in the first two rolls.

Math Maturity Food for Thought. In the three-digit Try for High game, a strategy specifies how to use the outcome from one's first roll and then how to use the outcome from one's second roll. This can be thought of as a 2-step strategy.

You are familiar with math problems that can be solved in one step, and the greater challenge students face as they work to solve 2-step, 3-step, and still more demanding problems. Here are three questions to think about:

1. What do we actually mean when we are talking about a 2-step math problem or a 3-step math problem? What is a step? (Here is one way to think about this question.

Students are taught algorithms for solving certain types of problems. Carrying out an algorithm requires carrying out a sequence of steps.)

2. Is the number of steps required to solve a particular type of problem a good measure of the complexity, mental challenge, or math maturity level of the problem? Thus, is a problem that takes five steps to solve necessarily more complex and mentally challenging that one that takes three steps to solve?
3. Solving $2x + 3 = 13$ is a two-step problem that is fairly easy. Solving $(2/3)\,x + 1/2 = 3/4$ is the same type of two-step problem, but totally boggles many students. Discuss why you think this is the case.
4. In the 3-place value game being discussed, there is a first decision and then a second decision—two clear decisions-making steps. Compare and contrast **decision** and **step**. Compare and contrast **algorithm** and **strategy**.

More About Luck

Luck means different things to different people. In this short section we examine the idea of luck in games and in other situations that are not typically called games. To begin, consider two different types of dice-roll or coin-flip types of games:

1. Games of pure chance. For example, you flip a true coin, and the win/lose criterion is "Heads I win, tails I lose." A coin has no memory. Over the long run, you will win about half the time and lose about half the time. It is common for a person who happens to have a *winning streak* or *hot streak* to say, "I was lucky." Various lotteries provide examples of games of pure chance that a great many people routinely play.
2. Games that have both chance and strategy-based decision making. The 2-digit and 3-digit Try for High and Go for Low Place Value games are good examples. Many popular board games, card games, and online games such as Worlds of Warcraft have this characteristic.

The second situation is common both in many popular games and in everyday life. Here are two quotes that fit with the everyday events in one's life and making one's own luck as well as with games involving a combination of luck and skill.

> Lucius Seneca was a Roman philosopher and statesman during the mid-1st century CE. He said: "Luck is what happens when preparation meets opportunity."
>
> Thomas Jefferson (third president of the United States) said: "I am a great believer in luck, and I find the harder I work, the more I have of it

The suggestions in these two quotes is that one (perhaps by random happenstance) encounters a wide range of opportunities. With appropriate preparation, one can take advantage of the happenstances. This is often described as making one's own luck.

Math Maturity Food for Thought. Think about luck in your life. Identify situations in which you feel you were lucky and other situations in which you feel you were unlucky. Then divide these into three categories:

1. Situations in which chance (perhaps pure chance) dominated the outcome.

2. Situations in which your level of preparation was a dominant factor in the outcome.
3. Situations in which a combination of chance and preparation dominated in leading to the outcome.

As you analyze these situations, think about what your students think about luck and what you would like them to think about luck. What teaching techniques do you use to help your students learn that in many cases, one creates their own luck? You might find it quite interesting to carry on a whole class discussion on the topic of luck.

Final Remarks

The strategy-based place value games presented in this chapter illustrate how a game can have several different educational values. Careful record keeping and then analysis of the results is important in many disciplines such as business and science. Posing and testing strategies and making and testing conjectures are similar to posing and testing hypotheses in science and other disciplines. Investigations of relatively simple games can lead to quite challenging math problems. All of these activities lead to increases in math maturity.

Chapter 11: Word Problems Using Dominoes

You can use a set of dominoes to play various versions of games called Dominos. However, it is also possible to use dominoes to create a wide variety of word problems. This chapter presents some examples of such word problems.

The figure to the right is copied from http://www.kosbie.net/cmu/fall-08/15-100/handouts/hw4.html. It shows all 28 Double-6 Dominoes in a commercially-produced set.

Sets of Double-9 Dominoes and Double-12 Dominoes are commercially available. These larger sets can be used to develop more challenging versions of the various word problems presented in this chapter.

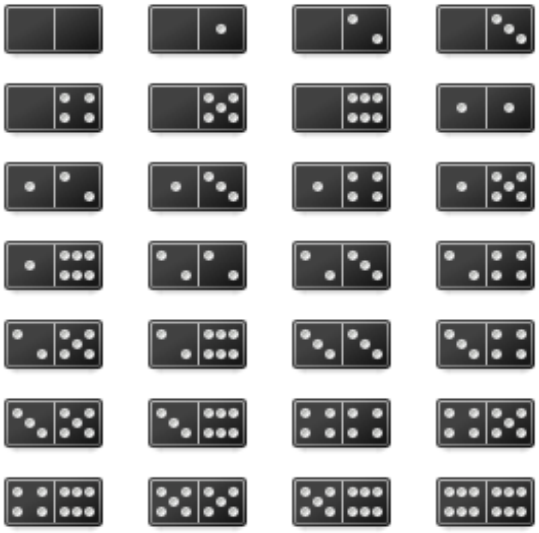


Figure 11.1 Complete set of Double-6 Dominoes.

Domino Games

Like dice and card games, domino games have a very long history. See http://www.pagat.com/tile/wdom/history.html. Today's domino tiles are rectangular solid-shaped tiles with pips on one side. Typically the width of a domino is half of its length. Young children often use them for building blocks. People also like to arrange the tiles standing on end in a manner so that as one falls over it knocks over the next, and so on. See http://en.wikipedia.org/wiki/Dominoes#Other_uses_of_dominoes.

Domino games can be thought of as math games. They vary in complexity and their cognitive challenges from quite simple to relatively complex. Many people—both children and adults—enjoy playing these games. See http://www.gamecabinet.com/rules/DominoGames.html.

This chapter is not about the traditional domino games. Rather, it presents a number of examples of using domino tiles in representing a variety of math word problems.

Math Maturity Food for Thought. As noted above, there are 28 different dominos in a set of Double-6-Dominos. Try doing a mental count—not looking at the picture given above or using pencil and paper—to verify that 28 is the correct count. In doing this, most likely you will mentally detect and make use of a pattern to aid in the counting process.

Now, do a mental count or calculation of the number of dominos in a Double-9 Domino set and in a Double-12 Domino set. One indicator of increasing math maturity is increasing ability to represent and solve math problems in one's "mind's eye." Eugene Maier's article (Maier, 1985) provides an excellent introduction to such visual thinking. The number of

dominoes in a double-*n* set is the (*n* + 1)st triangular number. Learn more about triangular numbers in Chapter 12.

Math manipulatives are one type of aid to helping students develop their math *mind's eye*. Think about other ways to help students increase their level of math maturity in representing and solving math problems making use of their mind's eye.

Some Simple Domino Word Problems

As you can see in Figure 11.1, each tile in a Double-6 Domino set has one side that is divided into two regions. Each region is either blank or contains a number of pips. Typically, the pips in a region are organized in the same manner as is used on the faces of a six-faced die (D6).

The number of pips in a region of a Double-6 Domino is a whole number in the range 0 to 6 inclusive. A blank (no pips) represents 0. In this chapter, we use the term "face" to designate one region of the pip side of a domino. Thus, we can think about the two faces of the pip side of a domino in much the same way as we think about the two faces produced by the roll of 2D6.

You might notice that if we ignore all of the dominoes in the Double-6 set that contain a blank face, then there are 21 remaining dominoes. Each of these dominoes represents a pair of numbers that can be produced by the roll of 2D6. If we use the two digits to make a 2-digit number, the smallest one is 11 and the largest is 66. However, we can't make all of the integers in [11, 66]. For example, we can't make 17, 18, or 19.

In some types of domino-based number activities, you might want to make the decision that a blank face is used to represent the number zero, since the face contains no pips (zero pips).

Math word problems for pre readers

Children can learn to play games involving rolling 2D6 before they learn to read. Here is a variation on such an activity. A young child is provided with the 21 dominoes in a Double-6 Domino set that does not contain a blank face. The child is given time to become familiar with different dominos and some vocabulary. A domino tile or a domino piece has two faces. The dots on a face are called pips.

Following oral instructions, the child is asked to respond to "word problems" (word questions) such as:

1. Find a domino that has 2 pips on one face and 1 pip on its other face. How many total pips does this domino have? (What is the sum of the number of pips on this domino's two faces?)
2. Find a domino that has 2 pips on each of its faces. How many pips does this domino have? This domino is called the "double 2s domino. Why do you think it has this name?
3. Find the double 3s domino. How many pips does this domino have?
4. Find a domino that has 5 pips on each of its faces. How many pips does this domino have?

5. Find a domino that has a total of 4 pips. (Note to teacher: This is an **Aha!** Moment. There are two different dominos that are solutions to this word problem.)
6. Find a domino that has a total of 5 pips. How many dominoes have a total of exactly 5 pips?
7. Find a domino that has a total of 13 pips. (Note to teacher. This is an **Aha!** Moment. There is no double-6 domino that has 13 pips. This word problem has no solution.)
8. Find the domino that has the smallest possible number of pips on its two faces. How many pips does this domino have?
9. Find the domino that has the largest number of pips. How many pips does it have?
10. How many dominos do you have? (Note to teacher: Can the child count to 21? Is this done is a systematic fashion so that no domino is counted twice and no domino is left out?)
13. How many of the dominos are doubles?
14. How many of the dominos are not doubles? (Note to teacher: An advanced student might answer this question by using subtraction or counting backward, combining the answers to the previous two questions.)

Here are a few variations on word problem activities given above.

- Have students work in groups (teams) of four, each group sharing one set of dominos. One student answers a question and the other group members decide whether it is a correct answer. Then they switch roles for the next question. In more complex situations, such as two or more different possible solutions, they can discuss the situation.
- The set of dominos is divided between two students working together. (Note that 21 is not evenly divisible by 2.) Now when a question is asked, each student attempts to find a solution within his or her collection of dominos. Sometimes only one student in a pair will have a solution. This is an **Aha!** moment. A student is faced by a math problem that has no solution in the available set of possible solutions. Sometimes both students in a pair will have a solution. Another **Aha!** moment—a problem with two or more different solutions.
- If your students know a little bit about subtraction, they can be given some subtraction problems. For example, find a domino where the larger face (the face with the larger number of pips) minus the smaller face (the face with the smaller number of pips) is the number 2. One solution is the (5, 3) domino with 5 pips on one face and 3 pips on the other face: $5 - 3 = 2$. Voila! Your turn: Find three more solutions.
- Students working in groups of two or more can make up problems to be solved by students in other groups or students in their group. (The general idea is to start with a domino that is to be a solution to the problem, and then make up a problem that has this domino as a solution.

- As students gain an understanding of the number zero, they can learn that a blank face on a domino has zero pips. Then students can do problems like those given earlier in this section, but using the complete set of 28 dominos in the Domino-6 Domino set.
- As student begin to learn to read, problems can be given to students in written form. As students begin to learn to write, they can write problems to be solved by others.

Math Maturity Food for Thought. Think about how students develop the erroneous idea that a math problem can have only one correct solution. Give some examples of math problems that have more than one solution and other math problems that have no solution. In doing this activity, specify the grade level at which you believe students can understand and solve such problems.

Then give some examples of problems outside of the discipline of math that might have more than one solution, or that might not have a solution. For example, think about how silly it sounds to talk about finding **the** solution to world hunger or **the** solution to homelessness.

Some More Complex Domino Word Problems

Figure 11.2 is a set of word problems based on the numbers represented by the pips on Double-6 Dominoes. These are more complex than the word problems given in the previous section.

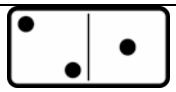
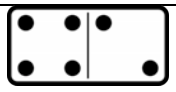
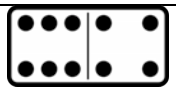
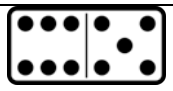
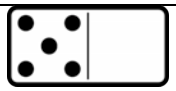
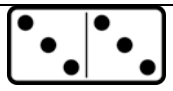
Problem	Solution
1. The sum of two whole numbers is 3. The difference (larger minus smaller) of the whole numbers is 1. What are the whole numbers? [Find the domino.]	
2. The sum of two whole numbers is 6. The difference (larger minus smaller) of the whole numbers is 2. What are the whole numbers? [Find the domino.]	
3. The sum of two whole numbers is 10. The difference (larger minus smaller) of the whole numbers is 2.What are the whole numbers? [Find the domino.]	
4. The sum of two whole numbers is 11. The difference (larger minus smaller) of the whole numbers is 1. What are the whole numbers? [Find the domino.]	
5. The sum of two whole numbers is 5. The difference (larger minus smaller) of the whole numbers is 5. What are the whole numbers? [Find the domino.]	
6. The sum of two whole numbers is 6. The difference (larger minus smaller) of the whole numbers is 0. What are the whole numbers? [Find the domino.]	
7. The sum of two whole numbers is 0. The difference of the whole numbers is 0. What are the whole numbers? [Find the domino.]	
8. The sum of two whole numbers is 8. The difference (larger minus smaller) is 3. What are the whole numbers? [Find the domino.]	no domino solution

Figure 11.2. A collection of sum and difference word problems.

In the above word problems, children with a low level of math knowledge may use the dominoes in a guess and check or exhaustive search manner. As their insights grow, they may solve such problems mentally and then search for a domino that matches their mental solution.

The problems given above can each be represented as a pair of simultaneous linear equations, as illustrated in Figure 11.3.

1. $x + y = 3$ $x - y = 1$	2. $x + y = 6$ $x - y = 2$	3. $x + y = 10$ $x - y = 2$	4. $x + y = 11$ $x - y = 1$
5. $x + y = 5$ $x - y = 5$	6. $x + y = 6$ $x - y = 0$	7. $x + y = 0$ $x - y = 0$	8. $x + y = 8$ $x - y = 3$

Figure 11.3. Algebraic equations representing the word problems in Figure 11.2.

Notice that the 8th set of equations in Figure 11.3 ($x + y = 8$ and $x - y = 3$) has a solution. The solution is $x = 5.5$ and $y = 2.5$. Alas, there are no dominoes with 5.5 pips on one face and 2.5 pips on the other face, so the solution is not in the set of dominos available to the student. We discuss this idea in the next section.

If the problems are given to students who have studied solving sets of linear equations in an algebra course, the students are quite apt to convert them into systems of linear equations and solve them using algebraic methods. However, students who have not yet encountered algebra can readily solve these problems.

Solution Space

In the domino-based word problems given above, students are directed to find a solution that is a domino from a specified set of dominoes. The specified set of dominoes is called a *solution space* or a *search space*. For problems based on the double-6 set of dominoes, the solution space consists of 28 dominoes. If we leave out dominoes with a blank face, the solution space consists of 21 dominoes.

The careful specification of a math problem includes a clear indication of the solution space. Often the statement of the problem makes the solution space clear even though it is not stated. For example, consider a problem involving finding the ages of two children. We "know" that ages of children are numbers larger than zero. That is implicitly assumed in such a problem.

In both of the types of domino word problems given so far in this chapter, a set of dominoes is the solution space. Each set of problems includes one problem in which there is no solution within the solution space.

Solution space is one of the big ideas in math. Suppose, for example, a child is learning how to count using positive integers, and you ask the child to divide a collection of seven toy cars into two equal collections. The problem has no solution within the solution space of positive whole numbers.

Math Maturity Food for Thought. The domino problems provide a way to introduce students to the idea of solution space and the idea that some problems do not have a solution within a specified solution space. Think back to your own math education. When and how did you learn about solution spaces? What are your current thoughts about what students should be learning about the idea of solution spaces?

Chapter 7 defines the term *problem* in terms of givens, goal, resources, and ownership. This definition does not make use of the term *solution space*. However, it does indicate that one has a clearly defined set of resources. Think about the following idea: The types of things that can be done with these resources constitute the solution space. Try this idea out on a variety of problems to see if it makes sense to you. This will help you to understand why it is so difficult to have clearly defined problems in areas such as eliminating hunger or homelessness, or ensuring sustainability of life on our planet.

More Complex Domino Word Problems

In the previous section, we presented word problems based on the sum and the difference of the two numbers on a domino tile. Of course, there are other ways to create problems using these two numbers. As an example, consider the sum of the two numbers and the product of the two numbers.

If the sum of the two numbers from a tile in a Double-6 Domino set is 9 and the product is 20, what are the two numbers? Find a tile whose two numbers are a solution to this problem.

Taking an algebraic approach, if we call the two numbers x and y, we have a pair of equations $x + y = 9$ and $xy = 20$. From an algebra point of view, we asking students to solve a pair of simultaneous equations, with one being linear and one nonlinear. This is an algebra 1 or algebra 2 problem.

However, the problem can be solved by students who have never encountered such higher math. The problem is initially stated in terms of the solution space being the double-6 dominoes. A student with a relatively low level of math maturity might use an exhaustive search strategy on this set of 28 dominos.

However, a little thinking can save a lot of work. What two positive whole numbers can be multiplied together to make 20? A little thought gives three possibilities: 1 20 = 20, 2 10 = 20 and 4 5 = 20. However, there is no domino in the double-6 domino set that has 10 or 20 for one of its faces Thus, the only possible solution is the domino with face values of 4 and 5. A quick mental check verifies that 4 + 5 = 9 and the problem is solved!

Here is another way to solve the above problem. What dominoes have faces that add up to 9? There are only two: the (3, 6) domino and the (4, 5) domino. The (3, 6) domino yields the product 3 6 = 18, not a solution. The (4, 5) domino is the one we want: 4 5 = 20. Voila!

Here is another "hard" problem. Find a domino in the double-6 domino set such that when the value of its larger face is divided by the value of its smaller face, the result is a prime number.

"Hmm" says the algebra student. "If I call the two numbers x and y, I can then write $x/y = a$ *prime number*. That doesn't look like a type of equation we have studied. Help!"

A non-algebra student might think: "I know that 2, 3, 5, 7, and so on are prime numbers. I need to find a double-6 domino where the quotient of its two faces is one of these numbers. Obviously 7 and larger primes are not possible answers, since the largest face in the domino set is 6. Hmm. How can a domino division lead to a 2, 3, or 5? Well, that doesn't seem like too hard a question. I see that 2/1 = 2, 4/2 = 2, and 6/3 = 2. Wow. I have already found three solutions to the problem. Now all I need to do is check to see if I can find dominos that give a quotient of 3 and/or dominos that give a quotient of 5. …"

How about the following problem. Find two double-6 dominoes such that the value of one of the dominos (total number of pips) times the value of the other domino (total number of pips) is 18.

The algebra student might think about a four-variable problem, with one domino having faces a and b, while the other has faces x and y. This leads to an equation $(a + b)(x + y) = 18$. "Hmm, says the algebra student. I have four variables and only one equation. Teacher! Help!"

Meanwhile, the non-algebra student thinks: "I need to find two dominoes, and their product must be 18. Well, I know that 1 18 = 18, 2 9 = 18, and 3 6 = 18. Let me think more about this. Let's see, how about 1 18 = 18? This means I need to find a domino with the sum of its two faces being 1, and another domino with the sum of its two faces being 18. Oops. There is no double-6 domino with 18 pips. Is there a solution based on 2 9 = 18? Aha! One domino could be the (1, 1) domino while the other is the (6, 3) domino. Solution: (1 + 1)(6 + 3) = 18. Alakazam! I just solved the problem. I wonder if there are other solutions? Is there another solution based on 2 9 = 18? Is there a solution based on 3 6? More than one solution?

Now try another target number. We like 24 because it has lots of **factor pairs**. You can find many solutions in the solution space of double-6 dominoes.

Factor pairs for 24: 1 24 = 24, 2 12 = 24, 3 8 = 24, 4 6 = 24

Try another target number. We suggest 36. Write all of the factor pairs of 36, and then find dominoes that have a total number of pips equal to each factor in a factor pair. For example: the (3, 1) domino and the (5, 4) domino: (3 + 1)(5 + 4) = 36. A-OK!

Try a target number for which there is no solution in the set of double-6 dominoes. An easy choice is 12 13 = 156. There is no domino with 13 pips in the double-6 set of dominoes. You cannot solve this problem in the solution space of double-6 dominoes, but you can solve it in the solution space of double-9 or double-12 of dominoes.

Math Maturity Food for Thought. The domino word problems given in this chapter can be solved by grade school students who have not studied algebra, unless the problem does not have a solution within the domino solution set. Students who are studying algebra may well view them as algebra problems. Using algebra, they will be able to solve some of the problems within the allowed domino solution set. They may also be able to solve other problems by using a larger solution set such as rational numbers or real numbers.

As a teacher of math, you want your students to learn to draw on the full range of math that they have studied. What can you do in your math teaching to continually give your students problem situations that require drawing on the full range of math that they have studied and are currently studying? How might this help increase their level of math maturity?

Problem Posing

Math problem posing (making up math problems) is a very important component of math and math maturity. With instruction and practice, students can gain in both skill and maturity in posing math problems. A recent Internet search on *math problem posing* produced nearly 50,000 hits. See, for example, http://www.cut-the-knot.org/Mset99/posing.shtml. One indicator of an increasing level of math maturity is an increasing ability to pose math problems and then solve or explore possible ways to try to solve the resulting problems.

The general idea of problem posing is simple enough. The idea is one of looking at possible problem situations and posing math problems that relate to the situation. For example, perhaps you workout on a treadmill about four times a week, walking or jogging about two miles in each workout. Problem: Estimate how many weeks of this level of activity it would take you to travel a distance equal to the distance across your state, across the United States, or around the world at the equator? Can you solve this problem mentally while on the treadmill?

Here is another example. You see a child playing with domino tiles. Each tile is a rectangular solid, but with somewhat rounded corners and edges. Let's use the words length, width, and thickness to refer to the longest edge length, the second longest edge length, and the third longest edge length. Here we are assuming that the length is twice the width, and that the thickness is quite a bit less than the width.

Now imagine in your mind's eye the child making a domino fence with pip sides down, and the fence enclosing a rectangular area. Figure 11.4 shows three examples.

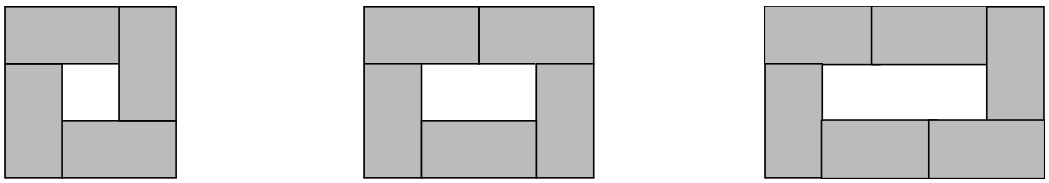


Figure 11.4. Three fenced in areas.

Notice that four dominoes can fence in an area of 1/2 of a tile, five dominoes can fence in an area of 1 domino, and six tiles can fence in an area of 1 ½ domino. Do you think this pattern continues?

This example is a "nice" math problem. You detect a pattern of the fenced area increasing by 1/2 domino for each domino you add. You then attempt to verify this pattern, perhaps first by construction of more examples. If the pattern continues to hold, you then try to make a proof that the pattern continues to hold for larger numbers of dominos.

Notice also that our unit of measure of area is 1/2 domino. This works fine even if different students have different sizes of dominoes, provided only that they all have dominoes whose length is twice their width.

Here are some problems that one can pose about this situation of using a set of double-6 dominoes to fence in a rectangular area. In these word problems, the fenced in area must be larger than zero. In all cases, a fence is to be rectangular and the fenced-in area varies in length and is 1/2 domino in width.

1. Figure 11.4 shows three fences made from 4, 5, and 6 dominoes, respectively. The areas enclosed increase by 1/2 for each additional domino. Explain how to use 7 dominoes to make a fence that encloses exactly 2 tiles, then 8 dominoes to make a fence that encloses 2 ½ dominoes, and so on. With this approach to fence building, what will be the area of the fenced-in region when all 28 dominoes are used?
2. Figure 11.4 shows a fence made from six dominoes that encloses an area of three half-dominoes (1 ½ dominoes). The enclosed area is a rectangle that is not a square. However, six dominoes can enclose a rectangle that is a square, and the square has an area of four ½ dominoes (2 tiles). Show how to do this.
3. From Figure 11.4 and Problem 2 given above, you see that it is possible to make fences that enclose square areas. Is it possible to make a fence that encloses a square that has an area of three dominoes? Is it possible to make a fence that encloses a square with an area of four dominoes? Using no more than 28 dominoes of fencing, what is the largest square area that can be enclosed?
4. What are the largest and smallest areas that can be enclosed using exactly nine dominoes? Answer the same question, but for 10 dominoes.
5. Figure 11.5 shows two non-rectangular fences. Each is made from eight dominoes, but one enclosed more area than the other. Can eight tiles enclose still more area? Explore this differing areas phenomenon for other numbers of tiles.

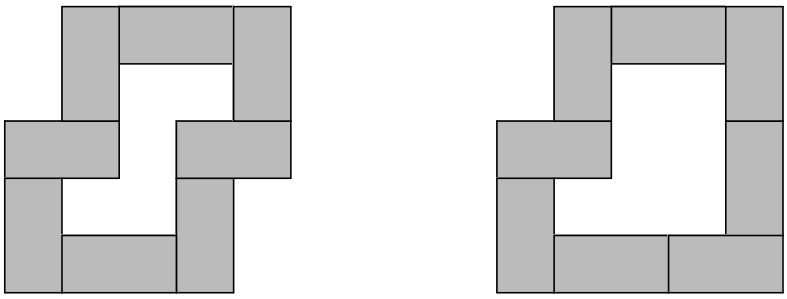


Figure 11.5. Two non-rectangular 8-tile fences.

Math Maturity Food for Thought. You can explore the fence problems using graph paper. Two adjacent squares on the graph paper serve as a graph paper model for a domino. Similarly, you can use computer graphics software to do the same types of exploration on a computer screen.

What are some advantages and disadvantages of each of these approaches? More generally, what are some learning advantages and problem-solving advantages that a student gains by working with several different representations of a problem?

Student Problem-posing in a Double-6 Domino Environment

This is an activity (a type of problem-posing game) with two players and one set of double-6 dominoes. One player is initially designated as the problem poser and the other as the problem solver. They switch roles after an agreed on number of turns.

The set of dominoes is placed between the two players, with pip sides up. The problem poser mentally selects a tile, makes up a math problem that has this domino as a solution, and writes the problem on a piece of paper. The problem poser then gives the problem to the problem solver whose task is to read the problem and then find a domino or dominoes that are solutions.

For example, suppose that the problem poser mentally selects the double-4s tile as a solution. Here are some examples of problems that the problem poser might pose:

1. Find a domino such that the sum of the two faces is eight and the difference of the two faces is 0.

   Notice that this is problem is not very carefully stated, because it does not say which of the two faces of the domino is to be subtracted from which. However, the problem solver is likely to be able to figure out that to get a difference of 0, the two numbers need to be the same. After that, a little thought or some guess and check leads to the (only possible) answer of the double-4 tile.

   This example helps to illustrate that it can be difficult to pose a word problem that has no ambiguity. One of the tasks a problem solver faces is to figure out of the problem is clearly stated with no ambiguity and/or to resolve ambiguities that are in the problem. Detecting and resolving ambiguities is an indicator of an increasing level of math maturity.

2. Find a domino such that the sum of the two faces is 8 and the product of the two faces is 15.

One approach might be to think about the sums 2 + 6 = 8, 3 + 5 = 8, 4 + 4 = 8, 5 + 3, and 6 + 2. Then test each of these to see which (if any) satisfy the requirement that the product of the two digits is 15.

A student with more math maturity will note that only the sums 2 + 6 = 8, 3 + 5 = 8, and 4 + 4 = 8 need to be considered.

3. Find a domino such that the quotient of the two digits is 1.

   This is another ambiguous problem statement, since it does not indicate which of the two faces to use as the divisor and which to use as the dividend. However, the problem solver will resolve this ambiguity through use of the knowledge that for the quotient of two integers to be 1, the two integers must be the same and they cannot be 0. This problem has 6 solutions from the double-6-domino set. Why is the double-blank domino (0, 0) not a solution?

The problem poser need not be restricted to just creating 1-domino problems. For example, "Find two dominoes whose four faces add to 13." One of the goals in having students practice on domino problems is for students to first solve the problem mentally (without using dominoes or pencil and paper) and then find dominoes that are solutions. In this example, a student might note that 1 + 12 = 13 and then think about whether there is a domino whose faces add to 1 and a domino whose faces add to 12. The student might continue by noting that 2 + 11 = 13, and thinking about whether there is a 2 domino (a domino whose faces add to 2) and an 11 domino (a domino whose faces add to 11). This line of thinking produces a number of answers to the problem.

The problem solver may well solve a problem without the use of dominoes, and then proudly select and show the solution as a domino or two. To carry this type of mental activity a step or two or more further, the problem poser need not use dominoes. The problem poser thinks of a problem, a solution space for the problem, and whether the problem has a solution in the solution space. The solution space may be a set of dominoes, but it might be something else such as the set of positive integers or the set of positive fractions. But that is another story for another time.

However presented, the problem and solution space are then written down and passed to the problem solver. The problem solver's goal is to find a solution in the solution space, or assert that there is no solution. If the assertion is that there is no solution, the problem solver must provide a good argument that there is no solution. Here are two examples:

1. The solution space is the set of positive even integers. Find two numbers whose product is 15. This problem has no solution. One possible argument is that 15 is an odd integer, and that the product of any two even integers is an even integer.

2. The solution space is the set of positive integers that are greater than 1. Find two integers whose product is 19. This problem has no solution. One possible argument is that 19 is a prime number whose only factors are 1 and 19.

Math Maturity Food for Thought. What are your personal thoughts and insights into having students generate problems to be solved by themselves and others? To what extent do the math books you teach from emphasize this topic? To what extent do you emphasize it in your teaching?

Also think about having students pose non-math problems to be addressed by themselves and others. For example, in a writing assignment, you might ask students to pose a problem that is important to students in the school (perhaps playground safety or need for more playground equipment) and then explore possible solutions in their paper.

Final Remarks

Dominoes, dice, cards, spinners, etc. can be used as math manipulatives. This chapter explores use of dominoes to represent and explore a variety of problems.

Problem posing (creating problems) is a very important aspect of the discipline of math and of an increasing level of math maturity. Dominoes provide a good environment for both teachers and students to practice problem posing.

Chapter 12: Factor Monster

Factor Monster is our name for a classic game about natural numbers, factors, proper factors, prime numbers, and composite numbers. You can play Factor Monster as a solitaire game or as a two-person game where one person is the player and the other person plays the role of Factor Monster.

Factor Monster is also called Taxman or the Factor Game. See Moniot (2/7/2007) and NCTM Illuminations (n.d.).

Taxman Game: http://www.maa.org/mathhorizons/pdfs/feb_2007_Moniot.pdf

NCTM Illuminations: Factor Game
http://illuminations.nctm.org/tools/factor/index.html

How To Play Factor Monster

Start with a list of **natural numbers** 1 to *n*. The game involves working with this set of integers. A small value of *n* makes the game easy, and a larger value of *n* makes the game more challenging. Let's use a small value, $n = 6$, to illustrate the game.

Here is the first part of the game rules. In playing the game with the natural numbers 1, 2, 3, 4, 5, 6 you will select a natural number that has not yet been used and that has one or more **proper factors** remaining in the list. A proper factor of a number is a factor that is less than the number. See Figure 12.1.

Number	Factor(s)	Proper factor(s)	Comments
1	1	none	1 is neither a prime number nor a composite number.
2	1, 2	1	2 is a prime number.
3	1, 3	1	3 is a prime number.
4	1, 2, 4	1, 2	4 is a composite number.
5	1, 5	1	5 is a prime number.
6	1, 2, 3, 6	1, 2, 3	6 is a composite number.

Figure 12.1. Examples of prime and composite numbers.

In your first move, you cannot take 1 because 1 does not have a proper factor. The only factor of 1 is itself. In the list of natural numbers 1 to 6, for your first move you may take 2 or 3 or 4 or 5 or 6.

Here is the next part of the game rules. After you select your number, alas, alack, and oh heck, greedy Factor Monster gets **all of the proper factors of the number you took**.

- If you take 2, Factor Monster gets 1, the only proper factor of 2.
- If you take 3, Factor Monster gets 1, the only proper factor of 3.
- If you take 4, Factor Monster gets 1 and 2, the two proper factors of 4.
- If you take 5, Factor Monster gets 1, the only proper factor of 5.

- If you take 6, Factor Monster gets 1, 2, and 3, the three proper factors of 6.

Finally, here are the rest of the rules. The number that you took and the proper factors that Factor Monster gobbles up are removed from the list. Then it is your turn again. You may take another number only if it has proper factors in the new list. More bad news! If the list contains only numbers that have **no** proper factors remaining in the list, then Factor Monster gets those numbers and the game is over. The player and Factor Monster add the numbers they have collected. The winner is the one who has the larger total.

Sample Game #1. Here's a game that Bob played with the list of natural numbers 1 to 6.			
	The List	**Bob**	**Factor Monster**
Starting list of natural numbers 1 to 6:	1, 2, 3, 4, 5, 6		
Bob takes 6 and removes it from the list.	1, 2, 3, 4, 5	6	
Factor Monster gleefully takes all of the proper factors of 6 and removes them from the list.	4, 5		1, 2, 3
Bob's turn. Oops! There are no remaining proper factors of 4 or 5, the two remaining numbers in the list, so Bob cannot take a number and Factor Monster gets 4 and 5.			4, 5
Bob and Factor Monster add their numbers.		6	15

Figure 12.2. A sample Factor Monster (FM) game in which the FM wins.

Hint to reader: When playing with paper and pencil, circle the numbers that the player takes, and then cross out the numbers that FM gets. Another approach is to play with numbered tiles, squares or circles. As play progresses, remove tiles and place them in the player's pile or Factor Monster's (FM's) pile. See a Blackline master in Appendix 1.

Okay, Bob has played one game and lost. He needs to start thinking about a strategy that might help him do better next time. Starting with the 6 proved to be a poor strategy. Bob did not think ahead about the consequences of this choice.

Looking for a Better Strategy

Now, back to the game. Here goes Bob with his second game using the list: 1–6. Bob has learned from experience to not take 6 as his first number, and so he tries a different starting number. See Figure 12.3.

Sample Game #2	**The List**	**Bob**	**Factor Monster**
Starting list of natural numbers 1 to 6:	1 2 3 4 5 6		
Bob takes 4 and removes it from the list.	1 2 3 5 6	4	
Factor Monster chomps the proper factors of 4.	3 5 6		1, 2
Bob takes 6, the only number that has a proper factor in the list.	3 5	6	
Factor Monster slurps up the proper factor of 6.	5		3

Bob's turn. But, there is no proper factor of 5 (Bob's only choice), so FM gleefully snarfs 5.			5
Bob and Factor Monster add their numbers.		10	11

Figure 12.3. Sample Factor Monster game # 2.

Bob did better in this game, losing by a score of 10 to 11. Maybe this is the best that Bob can do. Perhaps the game is such that the first player always loses?

Since Bob has only five possible first moves, and the number of possible moves he has after that is still more restricted, it is possible to try out all possible combinations. Eventually you will find that Bob can indeed win. But, what's the fun in taking an exhaustive search—try all possibilities—approach to solving the problem? Perhaps there is a better way?

Developing Strategies

Developing strategies is a fundamental aspect of problem solving. Here are some things to think about:

1. If there are a variety of versions of a problem, it is likely worth the effort to develop a strategy that is helpful in attacking all or a large percentage of the problems. Indeed, our formal education system teaches strategies for attacking frequently occurring types of problems that it deems important. Addition and multiplication of integers provide good examples of this.
2. There are some general-purpose strategies that are useful over a quite wide range of problems. *Exhaustive search* and *breaking complex problems into a collection of less complex problems* are two good examples. *Look it up* (using a physical or virtual library) and *asking a person with expertise in the problem area* are two more important strategies.
3. Through study and practice a person can get better at developing strategies.
4. Through practice, one can gain speed and accuracy in following the rules of a game (carrying out the procedures inherent to the game).

When you help your students learn to play a game such as Factor Monster, keep in mind that the game is an environment for learning. You have multiple goals in using this teaching environment.

While one of the goals is to learn to play the game, this is of less importance relative to the goals of learning than to develop and test strategies, and to having the opportunity for reflection and metacognition as one attacks a new learning task. Indeed, this strategies-related goal is more important than gaining additional practice in factoring and in using the vocabulary inherent to this game.

In playing Factor Monster, you get to make the first move. You might take the largest possible number, the smallest possible number, the largest possible prime number, the smallest possible prime number, the largest possible composite number, the smallest possible composite number, and so on.

Typically a beginner plays the game several times, perhaps giving little thought to what move to make first. This *little thinking* approach might be tried out with different sizes of n.

After playing a number of times, perhaps a pattern begins to emerge. Bob lost the $n = 6$ game when his first choice was 6. You might play the $n = 5$ game, start with a first choice of 5 and see what happens. You might play the $n = 7$ game, start with a first move of 7, and see what happens.

We are not going to tell you what happens in starting with n as the first move in the $n = 5$ or $n = 7$ game. That takes way the fun (and work) of testing and in increasing your skills in playing the game. However, suppose that you think you have discovered a pattern. Since you cannot test this pattern for every possible n, you need to develop a mathematical argument that your pattern is correct.

A great deal of the fun, joy, and satisfaction in learning math comes from attempting to find patterns and developing arguments that a pattern you have found is correct. In some sense, that is what math is all about!

Maximum Possible Score a Player Can Make

The table in Figure 12.4 shows the maximum possible score that the player can make for Factor Monster games with lists of numbers (value of n) less than or equal to 8. Notice that when $n = 1$, the player cannot take a number (no proper factor), so Factor Monster wins 1 to 0. For $n = 3$, the best the player can do is a tie.]

You can give your students the task of playing the game for these values of n, and trying to achieve the maximum score given in the figure. In this activity, students are given a quite specific goal.

You can also give them the task of trying to determine the maximum scores that can be achieved for higher values of n. This is a more difficult task. If a student defeats Factor Monster in a game, this does not prove that his or her score is the highest possible for this value of n.

List of natural numbers	**Player's maximum**	**In this case, Factor Monster then gets**	**Sum of scores**
1	0	1	1
1 2	2	1	3
1 2 3	3	3	6
1 2 3 4	7	3	10
1 2 3 4 5	9	6	15
1 2 3 4 5 6	15	6	21
1 2 3 4 5 6 7	17	11	28
1 2 3 4 5 6 7 8	21	15	36
1 2 3 4 5 6 7 8 9			
1 2 3 4 5 6 7 8 9 10			
Et cetera, et cetera			

Figure 12.4. Maximum score the player can make.

Gift Giver

The game Gift Giver is the flip side of Factor Monster. Using the same rules as Factor Monster, you choose your numbers so that **Gift Receiver** gets the higher score. For a given list of numbers, try to minimize your score and maximize Gift Receiver's score. Figure 12.5 is a sample game using the list 1 to 4.

	The List	**Bob**	**Gift Receiver**
Starting list of natural numbers 1 to 4.	1 2 3 4		
Bob cannot take 1 because there is no proper factor of 1 in the list. He takes 2.	1 2 3 4	2	
Gift Receiver takes the one-and-only proper factor of 2.	3 4		1
Great! There are no proper factors of 3 or 4 in the list, so Gift Receiver gets 3 and 4.			3, 4
Bob and Gift Receiver add their numbers. Bob is happy that Gift Receiver got the highest possible score with this list of numbers 1 to 4.		2	8

Figure 12.5. Bob and Gift Receiver play the Gift Giver game.

Bob began by taking 2. Thus Gift Receiver got 1, the proper factor of 2. Aha! That left only 3 and 4 in the list with no proper factors, so Gift Receiver inherited them. Do you agree that Gift Receiver received the maximum possible number of points and Bob received the minimum possible number of points for the list 1 to 4? Remember, Bob could not begin by taking 1 because it had no proper factor in the list. Figure 12.6 shows another game using the list 1 to 6. Bob, having won the preceding game by taking 2 in his first turn, again begins with 2.

	The List	**Bob**	**Gift Receiver**
Starting list of natural numbers 1 to 6.	1 2 3 4 5 6		
Bob cannot take 1 because there is no proper factor of 1 in the list. He takes 2.	1 3 4 5 6	2	
Gift Receiver takes the one-and-only proper factor of 2.	3 4 5 6		1
The only number in the list with a proper factor is 6, so Bob must take 6.	3 4 5	6	
Gift Receiver gets 3, the proper factor of 6.			3
There are no proper factors in the list for the remaining numbers 4 and 5, so Gift Receiver gratefully receives them.			4, 5
The numbers are added. Bob is happy that Gift Receiver got a higher score, but wonders if he could have gotten a still lower score.		8	13

Figure 12.6. Bob and Gift Receiver play the Gift Giver game.

What do you think? Is there a way to play so that Gift Giver gets less than 8 and Gift Receiver gets more than 13? The table in Figure 12.7 shows the minimum possible score that Gift Giver can make for selected values of n. You can give your students the task of playing the

game for these values of *n*, and trying to achieve the minimum score given in the table. In this activity, students are given a quite specific goal.

You can also give them the task of trying to extend the table by determining the minimum scores that can be achieved by Gift Giver for other values of *n*. This is a more difficult task. If Gift Receiver receives more points in a game than Gift Giver, that does not prove that Gift Receiver's score is the highest possible for this value of *n*.

List of natural numbers	**Gift Giver's minimum**	**Gift Receiver's maximum**	**Sum of scores**
1 2 3	2	4	6
1 2 3 4	2	8	10
1 2 3 4 5	2	13	15
1 2 3 4 5 6			21
1 2 3 4 5 6 7			28
1 2 3 4 5 6 7 8	10	26	36
1 2 3 4 5 6 7 8 9			
Et cetera, et cetera			

Figure 12.7 Gift Giver's minimum score and Gift Receiver's maximum score.

Try for a Tie or Close to a Tie

As noted earlier, it is easy to determine the total number of points that are scored in a FM game. The table in Figure 12.8 shows the total number of points for $n = 1$ to $n = 10$.

n	The list	Total number of points
1	1	1
2	1 2	3
3	1 2 3	6
4	1 2 3 4	10
5	1 2 3 4 5	15
6	1 2 3 4 5 6	21
7	1 2 3 4 5 6 7	28
8	1 2 3 4 5 6 7 8	36
9	1 2 3 4 5 6 7 8 9	45
10	1 2 3 4 5 6 7 8 9 10	55

Figure 12.8 Lists of numbers 1 to 10 and total points in each list.

Maia and Joel played Factor Monster and Gift Giver, and suggest a new game that they call Try for a Tie. In this game the object is to get a tie or as close to a tie as possible. Figure 12.9 is a sample $n = 6$ game played by Joel against Factor Monster. The sum of the numbers 1 to 6 is 21. Joel would like to barely win this game by a score of 11 to 10.

	The List	**Joel**	**Factor Monster**
Starting list of natural numbers 1 to 6:	1 2 3 4 5 6		
Joel takes 5.	1 2 3 4 6	5	
Factor Monster gets the one-and-only proper factor of 5.	2 3 4 6		1
The only number in the list with a proper factor is 6, so Joel **must** take 6, exactly as he had planned.	2 3 4	6	
Factor Monster scoops up 2 and 3, the remaining proper factors of 6.	4		2, 3
There are no proper factors in the list for the remaining number 4, so Factor Monster grabs it.			4
Joel and Factor Monster add their numbers. Joel has achieved his goal of winning by a score of 11 to 10.		11	10

Figure 12.9 Joel succeeds in winning by a score of 11 to 10.

Not to be outdone by Joel, Maia decides that she will play Gift Giver with the list 1 to 6 and try to lose by the near-tie score of 10 to 11. Her game is shown in Figure 12.10.

	The List	**Maia**	**Gift Receiver**
Starting list of natural numbers 1 to 6.	1 2 3 4 5 6		
Maia takes 4.	1 2 3 5 6	4	
Gift Receiver gets 1 and 2, the proper factors of 4.	3 5 6		1, 2
Maia gleefully gloms onto 6.	3 5	6	
As Maia wished, Gift Receiver receives 3, the remaining proper factor of 6.	5		3
There are no proper factors in the list for the remaining number 5, so Gift Receiver gets it.			5
Maia and Gift Receiver add their numbers. Maia has achieved her goal of losing by a score of 10 to 11.		10	11

Figure 12.10 Maia tries to lose by the score of 10 to 11

The table in Figure 12.11 shows possible near-tie scores for selected values of n. You can give your students the task of playing the game for these values of n, and trying to achieve the score given for the player. In this activity, students are given a quite specific goal.

You can also give them the task of trying to determine the almost-a-tie scores that we have omitted from the table. This is a more difficult task.

List of natural numbers	Player's score	FM's or GR's score	Sum of scores
1 2 3	3	3	6
1 2 3 4	4	6	10
1 2 3 4	4	6	10
1 2 3 4 5 (See if you can get a 7 to 8 or 8 to 7 score.)			
1 2 3 4 5 6	11	10	21
1 2 3 4 5 6	10	11	21
1 2 3 4 5 6 7			
1 2 3 4 5 6 7 8			
1 2 3 4 5 6 7 8 9			
1 2 3 4 5 6 7 8 9 10			
Et cetera, et cetera			

Figure 12.11 Try for a Tie scores that are ties or near-ties.

A Triangular Numbers Interlude

In the Factor Monster, Gift Giver, and Try for a Tie games, we deal with the natural numbers 1, 2, 3, …, *n*. The total number of points in a game is the sum of the natural numbers 1 to *n*.

Perhaps you have heard of triangular numbers. Make a triangular-shaped figure using one dot in the first row, two dots in the second row, three dots in the third row, and so on. See examples in Figure 12.12. The figure illustrates the first four triangular numbers—1, 3, 6, and 10. For more about the math of triangular numbers, see http://en.wikipedia.org/wiki/Triangular_number.

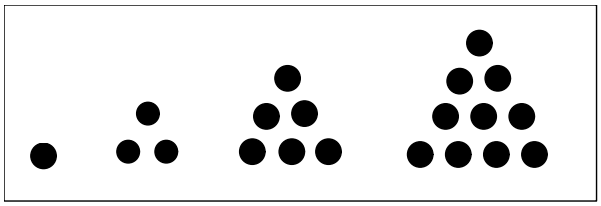


Figure 12.12. Dots used to illustrate triangular numbers.

You have probably seen a formula for the sum of the natural numbers 1, 2, 3, …, *n*. It is $(n)(n+1)/2$. To "see" where this formula comes from, examine the table in Figure 12.13.

Column 1: Numbers in ascending order	Column 2: Numbers in descending order	Column 3: Sums of pairs of numbers from first two columns
1	n	$n + 1$
2	n-1	$n + 1$
3	n-2	$n + 1$
4	n-3	$n + 1$
…	…	…
n-1	2	$n + 1$
n	1	$n + 1$

Figure 12.13. Adding the natural number $1 + 2 + 3 + \ldots + (n - 1) + n$.

Now notice that there are n numbers in the third column and each-and-every number is $n + 1$. Presto! The sum of the n numbers in the third column is $(n)(n + 1)$. But, each of the natural numbers 1, 2, … n is counted twice in forming this sum. So, the sum of the first n natural numbers is $(n)(n + 1)/2$.

Carl Friedrich Gauss was a mathematician sometimes called the "Prince of Mathematics", one of the people who helped create the math that we know and love. It is said that, as a student in school (we don't know his age at that time), his teacher gave his class the problem of summing the integers 1 to 100 in order to keep them busy so he could nip out for a beer. Oops! That may not be true, but it is a good story. Other students plodded through the task laboriously:

$1 + 2 = 3$

$3 + 3 = 6$

$6 + 4 = 10$

$10 + 5 = 15$

et cetera, et cetera

Gauss produced the answer in a few seconds, or perhaps a few tens of seconds by a process similar to the one shown in Figure 12.13. We encourage you to browse:

Carl Friedrich Gauss (http://en.wikipedia.org/wiki/Carl_Gauss)

The Math Forum (http://mathforum.org/library/drmath/view/57919.html)

Final Remarks

Factor Monster and variations listed in this chapter are excellent math education games. Players are engaged in a variety of activities that help increase their math maturity. Moreover, they face challenging problems and gain increased insight into the question, “What is mathematics.”

Chapter 13: The Game of Pig

Pig is a simple dice game that is lots of fun and chock-a-block full of learning and teaching possibilities. In the simplest form of Pig, each **turn** a player repeatedly rolls a die until either a 1 is rolled (resulting in a zero score for the turn) or the player holds (stops) and adds the **turn score** (sum of the rolls in the turn) to her or his **total score**. The game can end in various ways, such as:

- The game ends when a player reaches or exceeds an agreed-upon total score, such as 50 or 100. If there are two or more players, each player must enjoy the same number of turns. For example, if there are two players and the first player reaches or exceeds the target score, the second player can then try to tie or exceed the first player's score.
- The game ends after an agreed-upon number of turns have been played. We suggest 5 or more turns. Ten turns is good. In an *N*-turn game, every player gets—of course—*N* turns and has a fair shot at winning, the same as any other player who plays *N* turns.

If you have not previously encountered this game, you might want to skip down to the section titled **Sample Game** to get a concrete picture of the game. Then come back to here and continuing reading the background material.

According to http://en.wikipedia.org/wiki/Pig_%28dice%29, Pig was first described in print in 1945, but has a folk game history from long before then. Thus, there are variations in names and rules for the game. The simplest variation uses one six-faced die (1D6); a more complicated game uses two six-faced dice (2D6). Quoting from http://en.wikipedia.org/wiki/Pig_%28dice%29:

> Pig is one of a family of dice games called jeopardy dice games. … For jeopardy dice games, the dominant type of decision is whether or not to jeopardize previous gains by rolling for potential greater gains. Most jeopardy dice games can be further subdivided into two categories: jeopardy race games and jeopardy approach games. In jeopardy race games, the object is to be the first to meet or exceed a goal score (e.g. Pig, Pass the Pigs, Cosmic Wimpout, Can't Stop). In jeopardy approach games, the object is to most closely approach a goal score without exceeding it (e.g. Twenty-One, a die variant of Blackjack).

The Game of Pig is one of the units in the first year of Key Curriculum Press's *Interactive Math Program* (IMP). See http://www.keypress.com/x5436.xml. Quoting from *The Game of Pig Teacher's Guide*:

> Playing and Analyzing Pig involves students in a wide variety of mathematical activities. The basic problem for students is to find an optimum strategy for playing the game. In order to find a good strategy and prove that it is optimum, students work with the concept of expected value and develop a mathematical analysis for the game based on an area model of probability.

Beautiful! Let's teach students to design strategies for playing simple games, then more complex games, then games such as algebra, and hope that they learn how to create strategies beyond games, including the games of their lives.

Well, back to Pig. The rules of the game are simple. There are many strategies that can enhance how well one plays the game. Thus, the game is suitable for use over a wide range of grade levels and student abilities. An understanding of possible strategies, and exploration of these strategies, contributes substantially to an increase in math maturity.

Children who have introductory counting and addition skills can play the one-die version of Pig described below. While mental addition is emphasized, use of base-ten blocks or pencil and paper addition are also appropriate. Imagine base-ten blocks as a first-graders first "calculator." Players are encouraged to keep a written record of their play so they can analyze the results as they seek to develop good strategies.

The Basic Pig Game

Pig is played by one or more players. For the basic game, the only required equipment is one six-faced die (1D6) and a play sheet on which to keep track of the score as the game progresses. Figure 13.1 shows the first few lines of a play sheet that we use. See Appendix I for blackline masters of Pig play sheets.

Turn	**Die rolls this turn**	**Turn score**	**Total score**
			0
1			
2			
3			
…			

Figure 13.1 Pig Play Sheet

All players begin with a **total score** of zero (0). Players take turns rolling 1D6 one or more times and adding the outcome of each roll to their **turn score**. If the player rolls 1, he or she losses all of the points rolled during that turn, and the total score does not change. The good news: A player can stop rolling at any time, add her turn score to her total score, and pass the die to another player. Again, we suggest two ways to play Pig:

- Play an agreed-upon number of turns, say, *N* turns. Every player gets *N* turns. The player with the highest total score at the end of *N* turns is the winner. If there are two or more players, ties are possible. We like 5 to 10 turns.
- Play until one or more players reach a total score equal to or greater than a predetermined target number such as 50 or 100. If there are two or more players, play continues until each player has had the same number of turns. For example, if there are two players, and the player who rolled first reaches or exceeds the target number, then the second player can still try for a tie or a win.

Sample Game

Maia and Joel, whom you met in Chapter 10, see that Pig can be played by young students who might count the pips on the die in order to calculate their turn score, and then use base-10 blocks to add their turn score to their total score. Maia and Joel explore the game for possible use in their teaching.

To begin, Maia and Joel decide to play five turns. They agree to say aloud their potential new turn score after each die roll and add the outcome of the roll to their turn score on a play sheet. As they play using a combination of mental math and recording of scores, they are practicing a teaching method that they will use with their students.

Who goes first? They each roll the die, and the one who gets the lower score plays first. Joel rolls 3 and Maia rolls 4. Joel goes first. He sets his total score to zero and starts rolling the die. Table 13.2 shows Joel's first turn.

Joel rolls	He says aloud	He writes	His list of die rolls looks like this	Turn score	Total score
					0
5	5	5	5	5	
2	5 plus 2 is 7	+ 2	5 + 2	7	
3	7 plus 3 is 10	+ 3	5 + 2 + 3	10	
3	10 plus 3 is 13	+ 3	5 + 2 + 3 + 3	13	
4	13 plus 4 is 17	+ 4	5 + 2 + 3 + 3 + 4	17	
	I'll stop at 17	17	5 + 2 + 3 + 3 + 4	17	17

Figure 13.2. Joel's first turn.

Joel stopped at 17, added his turn score to his previous total score (0), and passed the die to Maia. As he rolled the die, Joel recorded the result of his rolls shown in Figure 13.2 (sum of die rolls, turn score, and total score) on his play sheet, shown in Figure 13.3.

Turn	Die rolls this turn	Turn score	Total score
			0
1	5 + 2 + 3 + 3 + 4	17	17

Figure 13.3 Joel's play sheet after his first turn.

Maia notes Joel's score of 17. She would like to roll up a first turn score at least as good as Joel's. Figure 13.4 shows her first turn.

Maia rolls	She says aloud	She writes	Her list of die rolls looks like this	Turn score	Total score
					0
3	3	3	3	3	
6	3 plus 6 is 9	+ 6	3 + 6	9	
3	9 plus 3 is 12	+ 3	3 + 6 + 3	12	
4	12 plus 4 is 16	+ 4	3 + 6 + 3 + 4	16	
	Maia's score is less than Joel's, so she decides to risk one more roll.				
1	Oops.	**1→0**	3 + 6 + 3 + 5 + 4 **1→0**	**0**	
Maia and Joel use the notation **1→0** if a 1 is rolled. Maia rolled 1, so she gets zero points this turn. She writes 0 as her turn score and new total score, and passes the die to Joel.				**0**	0

Figure 13.4. Maia's First Turn.

Maia recorded everything on her play sheet, and shares it with you in Figure 13.5.

Turn	Die rolls this turn	Turn score	Total score
			0
1	3 + 6 + 3 + 5 + 4 **1→0**	**0**	0

Figure 13.5. Maia's Turn 1 Play Sheet.

Maia and Joel played four more turns. Figure 13.6 shows the complete game. At the end of four turns, Joel was ahead 65 to 39, and so adopted a "be careful" strategy in his fifth turn. Maia decided to "go for a win"—keep rolling until she tied or won or rolled a 1 and lost. At the end of the 5-turn game, they copied the information from their play sheets and displayed it in Figure 13.6.

Turn	Die rolls for this turn	Joel		Maia	
		Turn score	Total score	Turn score	Total score
1 Joel	5 + 2 + 3 + 3 + 4	17	17		
1 Maia	3 + 6 + 3 + 4 **1 → 0**			**0**	0
2 Joel	4 + 5 + 5 **1 → 0**	**0**	17		
2 Maia	5 + 6 + 2 + 2 + 4			19	19
3 Joel	4 + 3 + 6 + 5	18	35		
3 Maia	6 + 6 + 5 + 3			20	39
4 Joel	3 + 6 + 5 + 2	16	51		
4 Maia	5 + 6 + 4 + 2 + 5 **1 → 0**			**0**	39
5 Joel	Joel has a good lead, so he is cautious and stops after 3 rolls. 6 + 3 + 5	14	65		
5 Maia	Maia needs 26 points to tie, 27 to win. As you see, she stopped at 26 and settled for a tie. 4 + 6 + 3 + 5 + 3 + 2 + 3			26	65
	Joel's and Maia's final scores		65		65

Figure 13.6. Completed game between Joel and Maia.

In their fifth (and last) turns, Joel and Maia each adopted a strategy. Joel was well ahead and adopted a cautious strategy of just trying to make a modest addition to his total score. Maia adopted a strategy of rolling until she at least tied, or rolled a 1 and lost.

Maia was lucky. She rolled seven times without rolling 1, and was quite happy to get a tie after being way behind. Notice that there is an advantage to being the last roller in the game of Pig. With knowledge of the final scores of the other players, the last player knows exactly how many points are needed to tie or win.

Strategies and Problem Solving

Earlier chapters have discussed the development and use of strategies. The discussion has been in terms of developing and using strategies for winning or playing well in a game. Here, we explore the topic a little more deeply.

Problem solving involves making decisions as one moves (or attempts to move) from a given initial situation toward a specified goal. A clearly defined problem has clearly defined rules (guidelines) and clearly defined resources. See Chapter 7 for more details.

Pig is a game. When played competitively, each player is faced by the problem of how to win the game. For each player, the game can end in a loss, a tie, or a win. Chance plays a major role in the game, so even the very best of players will lose from time to time. From this point of view, the problem a player faces is that of achieving a good win-lose-tie record.

Thus, the problem a player faces is that of developing a strategy or a set of strategies that will lead to a good record. This is an important idea. **Developing a strategy is a problem-solving**

task. Strategy-development problems are a type of problem that we want students to learn to solve.

In games that involve randomness (using dice, for example), an analysis of mathematical probability is apt to play a significant role in development of the strategies. Here are two examples illustrated in the Pig game played by Joel and Maia.

1. Both Joel and Maia need a strategy of when to stop rolling before they roll a 1 in each turn of the game. Both players know either 1) the number of turns in the game or 2) the target score. Both players record information about the rolls they have made so far during their turn. The second player also has information about the total score the first player made in his or her previous and current turns. An analysis of the probabilities is key to developing a good strategy.
2. As the game progresses, each player gains the information about how well she or he has done through the end of the previous turn. This information might be useful in developing a strategy for when to stop rolling in one's second turn, third turn, and so on. As illustrated in the sample game, this information allowed Maia to use a "go for a win or tie" strategy in her last turn. Notice that Maia did not do any analysis of probability in deciding on her strategy in the last round of the game. She needed to score 26 or more to tie or win. It took her seven rolls to get 26 points. The probability of Maia rolling seven times without rolling a 1 is $(5/6)^7 = 0.2791$ (about 28%). Maia was "lucky" to produce a sequence of seven rolls without rolling a 1.

Math Maturity Food for Thought. All teachers teach problem solving within the specific disciplines they teach. They help students learn strategies—and learn to develop strategies—relevant to solving important problems within these disciplines. All teachers can use this teaching/learning environment to teach for transfer of learning.

Each teacher faces the problem of what strategies to specifically teach and what emphasis to place on students learning to develop strategies. Analyze how you teach math. How much emphasis do you place on students memorizing strategies and developing speed and accuracy in using the strategies? How much emphasis do you place on students developing and testing their own strategies? Develop arguments for and against your current approach to strategies.

Pig is a Complex Strategy Game

In Pig, a player begins a sequence of rolls and needs to decide when to stop rolling and end his or her turn. Here are a few questions that this player might explore. In total, this set of questions suggests it is quite difficult to make optimal decisions in this game.

1. Suppose that in my current turn I have rolled n times (n being 1 or greater) and have not yet rolled a 1. Should I roll again or end my turn?
2. Suppose that in my current turn I have scored t points (t being 2 or greater). Should I roll again or end my turn?

3. Should I take into consideration the scores I have made in my previous turns and/or my total score (counting or not counting the potential turn score I have made so far in this turn)?
4. When and/or how should I take into consideration the current scores of my opponent or opponents?
5. How can I gain an advantage by figuring out some of the strategies being used by my opponents, and then taking this information into consideration in my own strategies?

Consider the first question. Suppose that you have rolled one or more times and have not yet rolled a 1. Should you roll again, or end your turn and pass the die to another player?

It turns out that this is a complex question. For example, suppose that this is the last turn of the game, and you are the last player to roll. Then you might decide to keep rolling until your total score is a tie with or better than the highest score among your opponents—or you roll a 1. If you stop before your tie or win or roll a 1, you will lose the game. If you keep on rolling, you have a chance to tie or win.

Suppose that this is the last turn of the game, but you are not the last player to roll. Maybe you are in the lead or a tie for the lead, but one or more players still has a turn coming up. Now you need to consider possible outcomes that might occur for these players as you make a decision about when to stop rolling.

Now suppose that you and another player are playing a set of ten games and the winner is the player who has the greatest total score in ten games. Because the second player has an advantage, you take turns being the second player. How does this game scenario affect your strategy?

Now suppose that the game you are playing is, say, the third game in a set of 10 games. The player with the greatest score in 10 games is the winner. How will this affect your strategy for this game if 1) you are ahead, 2) you are behind, or 3) you are tied?

And, of course, question 5 is always present. Are you playing against a novice whose strategy is mainly, “I go with what feels best to me at the moment.” Alternatively, perhaps your opponent always uses a fixed strategy such as, “In my turn, I always stop as soon as I have rolled four times or I stop when my turn score is 16 or more." Of course, this player may be forced to stop sooner, because of having rolled a 1.

Math Maturity Food for Thought. The questions about strategies given above are all applicable to the game of Pig. You might wonder whether similar questions are applicable to non-game situations. Before continuing your reading, see if you can come up with any examples in your life. Then read a few examples that occurred to your authors.

Have you ever skipped on sleep in order to do some extra studying for a test or to spend time reading a good book? Then the next evening arrives. Do you take into consideration the previous lack of sleep in making a decision for how to spend the evening?

Have you ever put of a major task (such as a term project or grading a set of papers) and ended up doing poorly because of lack f time? Did this learning opportunity affect your strategy the next time you were thinking about “putting off until tomorrow what you could do today?”

How about a similar idea, but over eating? Do you make use of a strategy of maintaining your current weight by under eating in the day following overeating? Or, has past experience led you to a different strategy?

In examples such as those given above, have you learned from observing and talking to others—adjusted your strategic decisions based on insight into the effectiveness of strategic decisions of others?

Strategy: Stop After a Fixed Number of Rolls

Suppose that a player decides to stop after a fixed number of rolls if 1 has not been rolled, and uses this strategy in every turn. Maia and Joel pondered this problem and made these notes:

- There are two types of turns: **nonzero turns** and **zero turns**. If your first roll is nonzero, and you stop before rolling a 1, you enjoy a **nonzero turn**. Add the turn score to your total score. If you roll again and you roll a 1, alas, you have a **zero turn**. Add zero to your total score—your new total score is the same as your old total score. Await your next turn with eager optimism.
- Strategy: stop after one roll if 1 was not rolled. This is a very low-risk strategy. Unfortunately, the turn score of a nonzero turn is a measly 2, 3, 4, 5, or 6. The mean turn score over many such nonzero turns is likely to be near 4, the mean of 2, 3, 4, 5, and 6. The theoretical mean nonzero turn score, also called the **expected value**, is 4. Of course, if you play many turns, you will roll a 1 (turn score = 0) on about 1/6 of the rolls, so your mean turn score for all turns, including zero turns, will most likely be less than 4.

  Maia and Joel have enough math knowhow to do a mathematical analysis of this situation. They came up with a theoretical mean turn score of 3.333… including both nonzero turns and zero turns. If you play 100 turns using this strategy, you might expect a total score of 100 (3.333) 333. Your authors simulated 10,000 turns using a TI-84 Graphing Calculator and got a mean turn score of 3.302. Hey! Close to the theoretical mean of 3.333.

- Strategy: stop after two rolls if 1 was not rolled. This is a low-risk strategy, but has more chance of getting a zero turn score than using the strategy of stopping after the first roll. The turn score of a nonzero turn is between 2 + 2 = 4 and 6 + 6 = 12. The mean of 4 and 12 is (4 + 12) / 2 = 8. If you think about this in terms of the theoretical mean nonzero score on the first roll plus the theoretical mean nonzero score on the second roll, you can see that the theoretical mean nonzero score for the two rolls (assuming neither is a 1) is 4 + 4 = 8.

Another way to analyze this situation is to make a list of all possible non-zero outcomes resulting from two die rolls. There are 25 of these, listed in Figure 13.7. Each sum of two rolls shows the first roll on the left and the second roll on the right. Therefore, 2 + 5 = 7 means that the first roll was 2 and the second was 5.

Nonzero turns: Roll 1 + Roll 2 = Sum of the Two Rolls					
2 + 2 = 4	2 + 3 = 5	2 + 4 = 6	2 + 5 = 7	2 + 6 = 8	Total = 30
3 + 2 = 5	3 + 3 = 6	3 + 4 = 7	3 + 5 = 8	3 + 6 = 9	Total = 35
4 + 2 = 6	4 + 3 = 7	4 + 4 = 8	4 + 5 = 9	4 + 6 = 10	Total = 40
5 + 2 = 7	5 + 3 = 8	5 + 4 = 9	5 + 5 = 10	5 + 6 = 11	Total = 45
6 + 2 = 8	6 + 3 = 9	6 + 4 = 10	6 + 5 = 11	6 + 6 = 12	Total = 50
Total = 30	Total = 35	Total = 40	Total = 45	Total = 50	**Total = 200**

Figure 13.7 The 25 nonzero turn outcomes using the stop after two rolls strategy.

The total of these 25 possible nonzero turn scores is 200; divide 200 by 25 and you get 8 as the **theoretical mean score** for a **nonzero turn**. Well, that is dandy, but what we want is the theoretical mean turn score for **all possible outcomes** of rolling a die twice, including nonzero turns and zero turns

There are 6 6 = 36 possible outcomes in rolling a die and then rolling it again. In Pig, using the stop after two rolls strategy, there are 25 **nonzero** outcomes, listed in figure 13.7. The other 11 outcomes result in a turn score of 0. The total score of a game is the sum of its nonzero turns, so the theoretical mean turn score is the total score of nonzero outcomes (200) divided by 36. Let's calculate:

(200 points in nonzero turns) / (36 possible outcomes) = 5.556 points per turn

If you play 100 turns, you might expect a total score of 100 (5.556) 556. Your authors simulated 10,000 turns using a TI-84 Graphing Calculator and got a mean turn score of 5.590, close to the theoretical value of 5.556.

You can see how this can go on for you and your students. You can extend Maia and Joel's notes shown above and ask questions such as the following.

1. Using the *stop after three rolls strategy,* what is the smallest possible nonzero turn score?
2. Using the *stop after three rolls strategy,* what is the greatest possible nonzero turn score?
3. In the *stop after three rolls strategy,* what is the mean of the smallest possible nonzero turn score and the greatest nonzero turn score?
4. In the *stop after three rolls strategy,* what is the theoretical mean value of a nonzero turn score?
5. In a *stop after N rolls strategy,* what is the smallest possible nonzero turn score?
6. In a stop after *N* rolls strategy, what is the greatest possible nonzero turn score?

7. In a *stop after N rolls strategy,* what is the mean of the smallest possible nonzero turn score and the greatest possible nonzero turn score?
8. In a *stop after N rolls strategy,* what is the theoretical mean value of a non-zero turn score?

Is there an optimal number of rolls before stopping? This question is easy to explore experimentally. Maia and Joel spent many happy hours rolling a **casino die** (the type of high quality die used in casinos) and recording data for 200 turns of each strategy: stopping after four rolls ($N = 4$), stopping after five rolls ($N = 5$), stopping after six rolls ($N = 6$), and stopping after seven rolls ($N = 7$). Figure 13.8 presents their experimental data.

	Stop after 4 rolls (N = 4)	**Stop after 5 rolls (N = 5)**	**Stop after 6 rolls (N = 6)**	**Stop after 7 rolls (N = 7)**
Number of turns	200	200	200	200
Total score	1517	1600	1643	1580
Mean turn score	7.585	8.000	8.215	7.900
Number of nonzero turns	95	81	69	57
Mean nonzero turn	15.96	19.75	23.81	27.72
Number of zero turns	105	119	131	143

Figure 13.8. Experimental data: 200 turns stopping after 4, 5, 6, and 7 Rolls.

The stop after six rolls strategy ($N = 6$) produced the highest mean turn score for 200 turns (8.215). This strategy produced 69 nonzero turns and 131 zero turns. Might this be a bit frustrating for an elementary school student to play with so many zero turns?

The stop after five rolls strategy ($N = 5$) was almost as good as the $N = 6$ strategy with a mean turn score of 8.000. This strategy had 81 nonzero turns and 119 zero turns.

Notice that the mean of nonzero turns for each strategy is very close to 4 times the number of rolls (N). For example, in the stop after four turns strategy, the mean nonzero turn was 15.96, very close to 4 x 4 = 16. Think about whether this is just a coincidence or if there is some mathematical reason behind this pattern.

Keep in mind that this is experimental data. If you and your students were to do this experiment, you would probably get somewhat different data.

In order to get more data, Maia and Joel used a TI-84 Graphing Calculator to simulate 10,000 turns of each strategy from $N = 1$ to $N = 7$. Their results are shown in Figure 13.9.

	N = 1	**N = 2**	**N = 3**	**N = 4**	**N = 5**	**N = 6**	**N = 7**
Total	33,024	55,900	68,937	76,963	79,050	80,535	79,381
Mean	3.302	5.590	6.894	7.696	7.905	8.054	7.938
Theoretical	3.333	5.556	6.944	7.716	8.038	8.038	7.814
% diff	0.93%	0.61%	0.72	0.26%	1.65%	0.20%	1.59%

Figure 13.9. TI-84 Simulation of 10,000 Turns for $N = 1, 2, 3, 4, 5, 6,$ and 7.

Figure 13.9 includes the percent difference of the simulated mean turn score and the theoretical mean turn score. This difference is less than 1% for $N = 1, 2, 3, 4$, and 7, and less than 2% for $N = 5$ and $N = 7$. Maia and Joel think that in a game of 10 turns, or a target score of 100, you probably won't notice much difference in strategies from $N = 4$ to $N = 7$.

Experimental data such as that shown in Figure 13.9 can be gathered by having a classroom of lab assistants (a.k.a. students) work in small teams to do experiments. The whole class results can be combined and analyzed in various ways, including by graphing or by computing the means of the means. Wow! How many topics in your math book are involved in this investigation? Math applied to the exploration of game strategies!

Strategy: Stopping After Reaching or Exceeding a Fixed Turn Score

In this section, we explore the strategy of stopping after a particular turn score has been reached or exceeded, or upon rolling a 1. As with the previous section, we ignore all other information, such as the scores of one's opponents and anything else going on in the game. The idea is to have a player choose a "stop at" turn score s and stick to that strategy in every turn throughout each game.

A player has rolled one or more times without getting a 1. The player has a total of t points scored so far in this turn. The player wants to have a strategy that is easy to remember, but has some logic or reasoning behind it. This strategy ignores information about scores of any other players in the game, or whether the game is nearing its end. Indeed, imagine a game environment to test strategies in which each player uses his or her strategy for many games, and no player knows the scores of other players.

Here is the situation. In the next roll, the player may roll a 1 and thus lose all t points rolled so far in this turn. Or, the player may roll a 2, 3, 4, 5, or 6. In this case, the player's turn score is increased and the player is again faced by the decision of whether to roll again or end the turn.

From a math probability point of view, there is one chance out of six of rolling a 1 and thereby scoring 0 in the turn. The t points scored so far in the turn are lost. The player has one chance out of six of gaining 2 points, one chance out of six of gaining 3 points, one chance out of six of gaining 4 points, and so on. In summary:

- Roll 1 and lose the turn points in this turn. The turn score is 0 and the turn ends. The probability of rolling 1 is one in six (1/6).
- Roll 2, 3, 4, 5, or 6 and gain that many points in this turn. The player has five chances out of six of the player rolling "not 1" and increasing her or his turn score. The probability of "not 1" is 5/6.

As the player's turn proceeds with "not 1" rolls and increases his or her turn score, the more points are at risk of being lost in the next roll. At some point, the player wisely decides to stop the turn with a positive turn score and add it to her or his total score.

Let's take an experimental approach in a hare and tortoise game – thanks, Aesop. Maia and Joel will play with Maia playing the role of the tortoise and Joel playing the role of the hare. They will play a 5-turn game. In every turn, Maia will use the strategy of stopping if her turn score t is 5 or greater, and Joel will use the strategy of stopping if his turn score t is 15 or greater. Who will win? Figure 13.10 shows the results from their first game.

Turn	Maia's rolls	Maia's turn score	Maia's total score	Joel's rolls	Joel's turn score	Joel's total score
			0			0
1	2, 4	6	6	5, 4, 5, 1	0	0
2	5	5	11	3, 3, 2, 5, 3	16	16
3	3, 6	9	20	1	0	16
4	6	6	26	5, 6, 1	0	16
5	5	5	31	2, 4, 2, 3, 3, 1	0	0
	Maia's mean score = 6.2			Joel's mean score = 3.2		

Figure 13.10. First game of Maia's strategy versus Joel's strategy.

As you can see, in this game "slow but steady" won the race. The tortoise prevails. Joel had one high-scoring turn, but scored 0 in all of his other turns. Undaunted, Joel is confident that he has the better strategy. Figure 13.11 shows their second game.

Turn	Maia's rolls	Maia's turn score	Maia's total score	Joel's rolls	Joel's turn score	Joel's total score
1	6	6	6	5, 5, 1	0	0
2	3, 1	0	6	5, 4, 4, 6	19	19
3	2, 3	5	11	3, 3, 5, 2, 1	0	19
4	3, 4	7	18	2, 2, 5, 6	15	34
5	4, 2	6	24	3, 2, 6, 5	16	50
	Maia's mean score = 4.8			Joel's mean score = 10		

Figure 13.11. Second game of Maia's strategy versus Joel's strategy.

In this game, slow but steady did not win. Although Joel had a couple of 0 scores, his three large scores ensured him a victory.

In the two games, Maia's mean scores per turn were 31 / 5 = 6.2 and 24 / 5 = 4.8. Joel's mean scores per turn were 16 / 5 = 3.2 and 50 / 5 = 10. Hmm ... what might Maia's mean score per turn might be if she plays a large number of turns using her strategy, and what might Joel's mean turn score be if he played a lot of turns using his strategy?

Maia knows that her strategy is less than wonderful, but she likes her stop at 5 or more strategy as an introductory strategy for, say, kindergarten and first-grade students. After using that strategy, young students can progress to stop at 6 or more, then stop at 7 or more, and then perhaps begin to ask about, and experiment with, stop at *n* or more, where they choose the value of *n*. In the game shown in Figures 13.10 and 13.11, Joel wanted to use stop at 20 or more, but Maia talked him down to stop at 15 or more.

After playing a few more games, they decided that it was more fun to collaborate and together roll 200 turns of each strategy. They also rolled 200 turns of the stop at 10 or more strategy. Figure 13.12 shows statistics from their experimental play tests of the stop at 5 or more strategy, the stop at 10 or more strategy, and the stop at 15 or more strategy.

	Stop if turn score ≥ 5	**Stop if turn score ≥ 10**	**Stop if turn score ≥ 15**
Number of turns	200	200	200
Total score	969	1492	1560
Mean turn score	4.85	7.46	7.80
Number of nonzero turns	145	129	93

Figure 13.12. Play test results from 200 turns each of three strategies

This experimental play test suggests that stop if turn score ≥ 15 is the best of the three strategies, with a mean turn score equal to 7.80. The stop if turn score ≥ 10 strategy is a close second place. As Maia suspected, her stop if turn score ≥ 5 yields a much smaller mean turn score of 4.85. She still likes this strategy for introducing Pig to kindergarten or first grade students.

Maia and Joel love to contrive questions for students to answer. Here are some questions inspired by the three strategies displayed in Figure 13.12.

- In the stop if turn score ≥ 5 strategy, what is the smallest possible nonzero turn score?
- In the stop if turn score ≥ 5 strategy, what is the greatest possible nonzero turn score?
- In the stop if turn score ≥ 5 strategy, what are the possible nonzero turn scores (the whole shebang—list all of them)?
- In the stop if turn score ≥ 10 strategy, what is the smallest possible nonzero turn score?
- In the stop if turn score ≥ 10 strategy, what is the greatest possible nonzero turn score?
- In the stop if turn score ≥ 10 strategy, what are the possible nonzero turn scores?
- In the stop if turn score ≥ 15 strategy, what is the smallest possible nonzero turn score?
- In the stop if turn score ≥ 15 strategy, what is the greatest possible nonzero turn score?
- In the stop if turn score ≥ 15 strategy, what are the possible nonzero turn scores?

Maia and Joel suggest that you and your math explorers investigate other "stop if" strategies:

Stop if turn score ≥ 18

Stop if turn score ≥ 20

Stop if turn score ≥ 23

Stop if turn score ≥ *N*. You choose the value of *N*.

Math Maturity Food for Thought. Can you think of any non-math examples that have some of the same characteristics as the strategies listed above? One that occurs to your authors is the

little kid who likes to snitch cookies from the family cookie jar. Probably nobody will notice that one cookie is missing. Indeed, perhaps two cookies will not be missed …

Variations

See http://cs.gettysburg.edu/projects/pig/piglinks.html for some variations. Here are some examples.

Pig Sneak Up on the Target Game. In this game, the object is to reach or most closely approach a target total score without exceeding it. The winners are players who have a total score equal to 100. If no player has a total score equal to 100, then the winners are players whose total scores are closest to 100, but not greater than 100. For example, suppose that the target score is 100, your total score is 97, and it is your turn. You may stop and keep your total score of 97 or you may roll again. If you roll 4, 5, or 6, you exceed 100 and lose. If you roll 1 or 2 and then stop, your new total score is 98 or 99 which might be good enough to win. If you roll 3, you are guaranteed a tie or a win.

- Pig can be played with a 4-faced die (D4), 8-faced die (D8), a 10-faced die (D10), a 12-faced die (D12), or a 20-faced die (D20), and with a die that you invent by writing numbers on the faces of a blank die. Use of a D4 decreases the arithmetic calculation challenges, while use of a D8, D10, D12, or D20 increases the mental arithmetic challenges. Of course, the die roll probabilities change when using these other dice. Hmm … Maia and Joel imagine a cornucopia of student projects using different dice.
- Two-Dice Pig is played with a pair of dice. A turn ends and no points are earned if just one of the dice comes up 1. If double 1s are rolled, a player gets no points for the turn and also loses all points scored in previous turns—the total score becomes 0. This is a double-jeopardy game!

Final Remarks

Pig and other jeopardy dice games are fun to play. They give players practice in doing arithmetic. But, the deeper learning occurs as students develop, test, and analyze strategies. An increasing level of skill in strategic math-related thinking and planning is an indication of increasing math maturity.

In Pig and other games we have analyzed, strategies can be tested experimentally and perhaps via mathematics. There are many math-related types of problems in which it is necessary to make decisions based mainly on experimental data that has been gathered in the past. Examples abound in medicine. We can gather data about drug interactions or bad reactions to various drugs. But, we don't know enough about the underlying science of drug uses in medicine to compute exact mathematical probabilities.

Chapter 14: More Games and Puzzles

> "It isn't enough just to learn—one must learn how to learn, how to learn without classrooms, without teachers, without textbooks. Learn, in short, how to think and analyze and decide and discover and create. (Michael Bassis; President, Westminister College in Utah, USA.)

This chapter provides brief introductions to games that are related to the types of games presented earlier in the book. It does not contain a detailed analysis of any of the games.

WordsWorth Variations

In Chapter 1 of this book, we introduced the game **WordsWorth Plus**, the first game in a copious collection of WordsWorth games. We begin this chapter with brief descriptions of variations of WordsWorth Plus and WordsWorth Times, a game that involves multiplication instead of addition.

WordsWorth is designed to help children increase their knowledge of both language and math. The game has many variations that make it useful over a very wide range of grade levels and student abilities, from grade 1, "up, up, and away" to high levels of learning. WordsWorth captures the flavor of what this book is about—helping students increase their levels of math maturity through use of math-oriented games and math word problems.

WordsWorth Plus Special Number Quest

Recall that letters are assigned letter values, as shown in Figure 14.1.

a = 1	b = 2	c = 3	d = 4	e = 5	f = 6	g = 7	h = 8	i = 9
j = 10	k = 11	l = 12	m = 13	n = 14	o = 15	p = 16	q = 17	r = 18
s = 19	t = 20	u = 21	v = 22	w = 23	x = 24	y = 25	z = 26	

Figure 14.1. Letter values for WordsWorth Plus

The WordsWorth Plus (WWP) of a word is the sum of the word's letter values. Here are some examples.

WWP (fun) = 6 + 21 + 14 = 41

WWP (game) = 7 + 1 + 13 + 5 = 26

WWP (learn) = 12 + 5 + 1 + 18 + 14 = 50

The WordsWorth Plus of a word is a natural number 1, 2, 3, 4, 5, and so on. There are many subsets of natural numbers that we think of as **special numbers**. Examples include the even natural numbers (often just called even numbers) 2, 4, 6, 8, et cetera, the odd natural numbers (often just called odd numbers) 1, 3, 5, 7, et cetera, and the prime numbers 2, 3, 5, 7, 11, and so on.

Variations of WordsWorth can make use of these special numbers. For example, one can set a goal of finding words whose WordsWorth Plus values are odd numbers 1, 3, 5, 7, 9, and so on.

Here are a few that we found—these words are in our favorite WordsWorth reference, the *Official Scrabble Players Dictionary*:

WWP (a) = 1 (Fortunately, the letter *a* is a word.)

WWP (ab) = 3 (abdominal muscle)

WWP (ad) = 5 (advertisement)

WWP (be) = 7 (the first person singular of the verb to be)

WWP (fab) = 9 (fabulous. The Beatles were called the "Fab Four.")

The only word that has a WordsWorth Plus equal to 1 is *a*. Are there other words that have a WWP equal to 3, 5, 7, or 9? The odd numbers go on and on and on, beyond the staying power of even the Energizer Bunny, so finding words that have a WWP equal to an odd number can go on as long as you like.

Here is a handy list of special numbers that we like. Of course, you can add lists of numbers that you think are special.

Odd numbers: 1, 3, 5, 7, 9, …

Even numbers: 2, 4, 6, 8, 10, …

Multiples of 3: 3, 6, 9, 12, 15, …

Multiples of n, where you choose n: n, $2n$, $3n$, $4n$, $5n$, …

Square numbers: 1, 4, 9, 16, 25, …

Cubic numbers: 1, 8, 27, 64, 125, …

Triangular numbers: 1, 3, 6, 10, 15, …

Prime numbers: 2, 3, 5, 7, 11, …

Composite numbers: 4, 6, 8, 9, 10, …

Powers of 2: 1, 2, 4, 8, 16, …

Powers of 3: 1, 3, 9, 27, 81, …

Factorial numbers: 1, 2, 6, 24, 120, …

Fibonacci numbers: 1, 2, 3, 5, 8, 13, …

Palindromic numbers such as: 11, 22, 121, 606, 2332, …

WordsWorth Plus Zero Quest

This variation uses a table of letter values that are integers, alternately positive and negative. The WordsWorth Plus of a word is a positive integer, a negative integer, or zero— the object of our quest. Here are the letter values.

a = 1	b = 2	c = 3	d = 4	e = 5	f = 6	g = 7	h = 8	i = 9
j = 10	k = 11	l = 12	m = 13	n = 14	o = 15	p = 16	q = 17	r = 18
s = 19	t = 20	u = 21	v = 22	w = 23	x = 24	y = 25	z = 26	

Figure 14.2. Letter values for WordsWorth Plus Zero Quest

Using the letter values in Figure 14.2, let's calculate the WordsWorth Pluses of *fun*, *game*, and *learn*, three of our favorite words.

WWP (fun) = 6 + 21 + (14) = 1

WWP (game) = 7 + 1 + 13 + 5 = 26

WWP (learn) = 12 + 5 + 1 + (18) + (14) = 38

The goal of WordsWorth Plus Zero Quest is to find words that have a WWP equal to zero, or close to zero. Here are some examples:

WWP (a) = 1	Close to 0. The **distance** of 1 from 0 is 1.
WWP (ab) = 1 + (2) = 1	Close to 0. The **distance** of 1 from 0 is 1.
WWP (baa) = 2 + 1 + 1 = 0	Aha! We found a zero word.
WWP (be) = 2 + (5) = 3	The **distance** of 3 from 0 is 3.
WWP (fab) = 6 + 1 + (2) = 7	Not close to 0. The **distance** of 7 from 0 is 7.

Imagine two kinds of Zero Quest games. One game is to find as many words as you can find that have a WWP equal to zero or close to zero. WWP (fun) = 1, which is close to zero. It is a **distance** of 1 from zero. Distance from zero is a handy measure to use in designing a scoring system for WWP Zero Quest. The numbers 1 and 1 are the same distance from 0, and get the same score, 1. The numbers 2 and 2 are the same distance from 0 and get the same score, 2. The distance from zero of a number is the **absolute value** of the number.

Another game might be to find, say, ten words whose total WWP (the sum of the ten WWPs) is equal to zero or close to zero. In a multiplayer game, the winner is the player whose sum of ten words has the smallest total distance from zero.

WordsWorth Times

The WordsWorth Times (WWT) of a word is the product of the word's letter values. The letter values, shown in Figure 14.3, are the same as in WordsWorth Plus.

a = 1	b = 2	c = 3	d = 4	e = 5	f = 6	g = 7	h = 8	i = 9
j = 10	k = 11	l = 12	m = 13	n = 14	o = 15	p = 16	q = 17	r = 18
s = 19	t = 20	u = 21	v = 22	w = 23	x = 24	y = 25	z = 26	

Figure 14.3. Letter values for WordsWorth Times

Here are some two-letter words that we hope students can do using mental arithmetic.

WWT (ah) = 1 8 = 8	The words *ah* and *ha* are **reverses** of each other, and have the same WWT because multiplication is a **commutative**

WWT (ha) = 8 1 = 8	**operation**. Words that are reverses of each other are **semordnilaps** (**palindromes** spelled backwards).
WWT (be) = 2 5 = 10 WWT (hi) = 8 9 = 72	We hope that students who are learning the times table can do these using mental arithmetic.

Let's calculate the WordsWorth Times of some 3-letter words. The object of this activity is to find the WordsWorth Times of a word, not to practice multiplying numbers, so a calculator is an appropriate tool for this task.

WWT (dad) = 4 1 4 = 16	The words dad and mom are **palindromes**.
WWT (mom) = 13 15 13 = 2535	
WWT (act) = 1 3 20 = 60 WWT (cat) = 3 1 20 = 60	The words *act* and *cat* are **anagrams** (**permutations**) of each other, and have the same WWT because multiplication is a **commutative operation**.
WWT (ate) = 1 20 5 = 100 WWT (eat) = 5 1 20 = 100 WWT (eta) = 5 20 1 = 100 WWT (tea) = 20 5 1 = 100	The words *ate*, *eat*, *eta*, and *tea* are **anagrams** (**permutations**) of one another, and have the same WWT because multiplication is a **commutative operation**. The words *ate* and *eta* are **semordnilaps**. We think that you can use semordnilaps to amaze and amuse your friends.

One can set a goal of finding three-letter words whose WordsWorth Times values are even, or odd, or prime, or another special number. Hmm. How can the product of three integers be a prime number? Easy—one of the integers must be a prime number and the other two integers must each be 1. The word *baa* is an example. WWT (baa) = 2 1 1 = 2. Can you find more examples? Hint: Try anagrams of *baa*.

Number Factory Games

Number Factory games are puzzles in which you use a set of numbers and a set of mathematical operations as "raw materials" to "manufacture" designated numbers.

In the Number Factory, your task is to use raw materials to manufacture **numerical expressions** that have **values** equal to designated numbers. The raw materials are:

Decimal digits: produced by rolling a set of dice. A game based on using a set of D6 will make use of the digits [1,6] while a game based on using a set of D10 will make use of digits [0,9].
Operations: addition, subtraction, multiplication, and division.
Parentheses: ( ).

In the Digit Factory game, a player uses some method to generate several digits. For example, using 5D6, a player might generate the digits 1, 2, 3, 3, and 6. The goal is then to combine the 5 digits using only the four arithmetic operations addition, subtraction, multiplication, and division to make as many different non-negative integers as possible. Use of parentheses is allowed.

Here are some examples. In these examples, we have not generated all possible integers that can be made from the five digits. Often there is more than one way to produce a particular integer. However, one's "score" is just the number of different integers produced.

Example 1. Roll 5D6 and get 1, 2, 3, 3, 6.

- (3 + 1 - 2)*6/3 = 4.
- 6*3/3 + 2 - 1 = 7. (Note use of 3/3 = 1.)
- (3 - 3)*6 + 2/1 = 2. (Note use of 3 - 3 = 0 and 0*6 = 0.)
- (6 - 2*3)/3 + 1 = 1.

Example 2. Roll 5D6 and get 1, 2, 3, 4, 5.

- (1 + 2 - 3)*4/5 = 0.
- (5 + 4 - 3)*1/2 = 3.

Example 3. Roll 5D6 and get 3, 3, 3, 3, 3.

- (3 + 3 - 3)*3/3 = 3. (Note use of 3/3 = 1.)
- (3 - 3)*(3 + 3)/3 = 0. (Note use of 3 - 3 = 0 and 0*any-number = 0.)
- (3 - 3) + 3*3*3 = 27

This game can be played with D4, D6, D8, and D10. The game is made simpler by using fewer dice; it is made more complex by using more dice. The game can be used as a one-player game. Two or more players or two or more teams can compete. (Each player or each team gets the same set of digits to work with.)

Here are three variations.

1. Drop the requirement that **all** of the digits be used in forming an integer.
2. Change the goal, so that it to create arithmetic expressions that have an integer value in the range [0, 10] or [0, 100].
3. Enlarge the number of types of operations that can be used. For example allow concatenation (for example, the two digits 2 and 5 can be concatenated to make 25 or 52) and exponentiation (for example, using exponentiation the two digits 2 and 5 can be combined to make $5^2 = 25$ or $2^5 = 32$).

The games listed above are closely related to the Four 4s Problem that many students encounter in school. Use four 4s to form as many of the integers [1, 100] as possible. See http://mathforum.org/ruth/four4s.puzzle.html.

HurkleQuest

> *The Hurkle is a happy beast that lives in another galaxy on a planet named Lirht that has three moons. Hurkles are favorite pets of the Gwik, the most intelligent denizens of Lirht.* (Paraphrased from "The Hurkle Is a Happy Beast," a story by Theodore Sturgeon, science fiction hall-of-fame author who lived in Eugene, Oregon.)

A long time ago, Bob (one of your authors) and his friend George (a dragon) invented a simple game called Hurkle and published the first primitive version of Hurkle as a computer game in *People's Computer Company*, April 1973. The game has evolved into a system of games called *HurkleQuest*, including number line games, two-dimensional games, and games played on maps of Oregon, the USA, and other geographical areas.

HurkleQuest on the Number Line is the simplest game. The *Hurkle Hider* hides Hurkle on the number line. Players guess Hurkle's hiding place, and receive "fuzzy feedback" clues. The clues are six colors of the rainbow: red, orange, yellow, green, blue, and violet.

> Hurkle is hiding at a whole number on the number line [0, 100]. If you are the Hurkle Hider, you can choose an upper limit on the number line that you like. After students have played their first game, we usually increase the limit to 200 or 300.

Each turn the player (you, the Hurkle seeker) makes three guesses. For example, if the first three guesses are 20, 50, and 80, you would submit your guesses as:

- Guess #1: 20
- Guess #2: 50
- Guess #3: 80

The Hurkle Hider will then give you a color clue for each of your guesses. Here are the possible clues:

- RED: hot, very close to Hurkle's hiding place.
- ORANGE: very warm, close to Hurkle's hiding place, but not as close as RED.
- YELLOW: warm, not as close as RED or ORANGE, but closer than GREEN.
- GREEN: cool, not as close as YELLOW, ORANGE, or RED, but closer than BLUE.
- BLUE: cold, far from Hurkle's hiding place, but closer than VIOLET.
- VIOLET: very cold, very far from Hurkle's hiding place.

The Hurkle Hider's clues are related in a nonlinear measure of the **distance** of a guess from Hurkle's hiding place, as shown in Figure 14.4.

Distance	Color		Distance described in words
1	red	very hot	very close to Hurkle's hiding place
2 or 3	orange	hot	close to Hurkle's hiding place
4 to 7	yellow	warm	farther than orange, closer than green
8 to 15	green	cool	farther than yellow, closer than blue
16 to 31	blue	cold	far from Hurkle's hiding place
32 or more	violet	very cold	very far from Hurkle's hiding place

Figure 14.4 HurkleQuest on the Number Line color clues.

This beginner's version of HurkleQuest on the Number Line integer [0, 100] usually takes four to eight turns to play. We recommend using 3-inch-wide adding machine tape to make big number lines. We also use a meter stick as a number line. Each centimeter mark is a possible Hurkle hiding place.

There are many variations of HurkleQuest on the Number Line games, and also two-dimensional games: HurkleQuest in Quadrant 1 and HurkleQuest in Quadrants 1, 2, 3, and 4. You can find much information about HurkleQuest, including stories about many games played by email, at Curriki. Go to http://www.curriki.org and search for **hurklequest**.

Name That Number

Name that Number games are simple dice-based games that give students practice in oral expression of multi-digit numbers. The games can be played with D4, D6, D8, or D10 (with faces labeled 0 through 9). If one is playing using a pair of dice, it is helpful if the two dice are of different colors or different sizes. An alternative to this is to roll one die at a time, keeping track of the first-rolled die and the second-rolled die. Similarly, in playing with three dice it is helpful to have three different colors or three different sizes.

Here are two simple games to be played with 2D10 (one green, one red) with the faces numbered integer [0, 9]. One can play similar games using three or more dice. Through practice, a student can gain considerable speed and accuracy in oral reading of dice.

1. Roll 2D10 and say the name of the number formed by the two digits in the order red green, and the two digit number formed by the two digits in the order green red. A roll of 7 and 0 would produce the two numbers 07 (which we call seven) and 70 (which we call seventy).
2. This is an extension of (1) above. Playing with a partner who serves both as a timekeeper and a referee (checking for correctness), see how many correctly answered rolls are completed in one minute. Notice that this creates a situation in which the referee might make a mistake. In athletic events such as basketball and football, the game clock is stopped while disagreements are being settled. This suggests one might want to make use of a stopwatch in the timing. Also, it might be desirable to have a player, a referee, and a timekeeper.

We have not specified who does the counting of the number of correctly answered rolls in the given time period. The referee might do this, the dice roller might do this, or both might do this. Notice the increased cognitive load on the dice roller if he or she has to both maintain a count of successfully processed roles and do the processing of each roll.

Cognitive load and cognitive overload are important areas of brain science research. A person can increase their cognitive capabilities by pushing themselves—by working on hard problem-solving tasks that are near the limits of their cognitive capabilities. See http://i-a-e.org/newsletters/IAE-Newsletter-2010-41.html.

Number Race 0 to 12 Game 1

In this section we give brief introductions to two different racetrack games that are played with 2D6 and a five-part racetrack such as the one illustrated in Figure 14.5. While the game can be played in a solitaire mode, two or more players often play it in a competitive mode. Each player has his or her own copy of the racetrack.

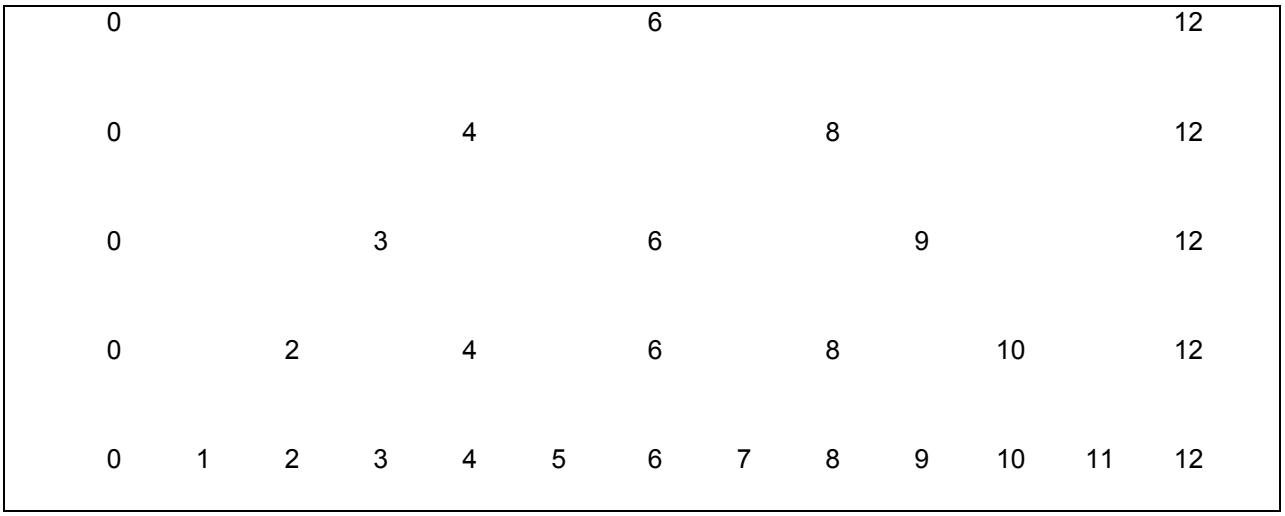


Figure 14.5. Five-part racetrack for Number Race 0 to 12.

In Number Race 0 to 12 Game 1, you roll 2D6 and use the outcome of the dice roll to move racers from 0 to 12 along the five racetracks shown in Figure 14.5. To begin the game, put a racer at 0 on each racetrack. The racer may be a figurine or anything that you fancy to represent your racer. The object of the game is to move all five racers to the finish line (12). In one round, every player gets a turn. The winner is the player who gets all five racers to 12 at the end of a round, so it is possible for two or more players to tie for a win.

A player's move in a turn is based on rolling 2D6 and making decisions based on three rules/options. To move your racers towards the finish line (12), roll 2D6 and use the outcome to move one or two racers from a numbered dot (●) to another numbered dot in any of these ways:

> Use the numbers on the **two dice** to move **two racers** along their tracks from one dot to another dot. If you roll **3** and **4**, you may move a racer on Track 3 or Track 5 a distance of 3 and also move a racer on Track 2, 4, or 5 a distance of 4. If you have a racer on 6 on Track 1, 3, 4, or 5 and roll a **6**, you may move that racer from 6 to 12 because 6 is the distance from 6 to 12. If a racer is on 8 on Track 2, you must roll a **4** to move it to 12. You may not use a roll of **5** or **6** to move a distance of 4.
>
> Add the numbers on the two dice and use the sum to move **one racer** that distance on its track. If you roll **6** and **6**, you may add them and use the sum (12) to move one racer all the way from 0 to 12.
>
> Subtract the numbers on the two dice and use the difference to move **one racer** that distance on its track. This can be handy near the end of a game. Suppose all of your racers have finished except the racer on track 5, which is at 10. You need a **2** to move that racer to 12. You roll 3 and 5, calculate 5 – 3 = 2 and move your racer to the finish line.

As the game proceeds, you may find that a particular roll does not provide any legitimate moves. A good strategy may decease the number of times that a roll does not advance any of your racers.

Number Race 0 to 12 Game 2

In Number Race Game 2, you roll 3D6 and use the outcome to move one, two, or three racers along their tracks from one numbered dot (●) to another numbered dot in any of these ways:

Use the numbers on the three dice to move three racers along their tracks.

Use one die to move one racer. Add or subtract the other two dice and use the sum or difference to move another racer.

Add and/or subtract the three dice and use the result to move one racer. For example, if you roll 3, 4, and 5, you can calculate 3 + 4 + 5 = 12 and move one racer from to 0 to 12.

Let *a*, *b*, and *c* represent the numbers on the three dice in any order. You may use the following algebraic alakazams to calculate moves for one or two racers:

Two-dice calculations using addition, subtraction, and multiplication: Use $a + b$, $a - b$, or $a \quad b$ to move one racer. Use the 3rd die to move another racer, if possible.

Three-dice calculations using addition, subtraction, and multiplication: Use $a + b + c$, $a + b - c$, $a - b - c$, $a \quad b + c$, or $a \quad b - c$, or $a \quad b \quad c$ to move one racer.

Three-dice calculations using addition, subtraction, multiplication, and parentheses: Use $a(b + c)$ or $a(b - c)$ to calculate a move for one racer. For example, if you roll 2, 2, and 4, you can calculate 2(2 + 4) = 12 and move one racer from 0 to 12. Hey! These calculations use the **distributive properties** of multiplication over addition and multiplication over subtraction.

Number Race Math Notes

Each Number Race 0 to 12 game has a five-part racetrack with the individual parts labeled Track 1 through Track 5. In each racetrack, the **minimum distance** between numbered dots (●) is a **factor of 12**. A legal move on any track is a **multiple** of the minimum distance on that track. To begin a game, put a racer at **0** on each track. To move a racer, roll 2D6 (Game 1) or 3D6 (Game 2).

Track 1. Minimum distance: 12 / 2 = 6. Possible moves: 6 and 12.

Track 2. Minimum distance: 12 / 3 = 4. Possible moves: 4, 8, and 12.

Track 3. Minimum distance: 12 / 4 = 3. Possible moves: 3, 6, 9, and 12.

Track 4. Minimum distance: 12 / 6 = 2. Possible moves: 2, 4, 6, 8, 10, and 12

Track 5. Minimum distance: 12 / 12 = 1. Possible moves: 1, 2, 3, 4, 5, …, 11, 12. This is the only track in which you can move your racer a distance that is an **odd number**. If your racer is stuck on 11 on Track 5 and you roll 5 and 4, calculate 5 – 4 = 1 and zoom that racer to the finish line.

In our experience, elementary school students who play Number Race games become quite adept at creating **algebraic expressions** that will move their racers toward the finish line.

KenKen

KenKen is a quite popular arithmetic puzzle. Many different Websites offer free puzzles you can play online or print. See, for example:

KenKen (n.d.). KenKen: *The puzzle that makes you smarter*. Retrieved 7/27/2010 from http://www.kenken.com/.

Mathdoku (n.d.). Retrieved 12/22/2010 from http://www.mathdoku.com/.

The Math Forum@ Drexel (n.d.). KenKen. Retrieved 11/5/2010 from http://mathforum.org/kenken/introduction.html. Quoting from the first of these two Websites:

> KenKen puzzles on the Math Forum are offered in the same format as our regular Problems of the Week. You can find all the puzzles at http://mathforum.org/kenken. The puzzles are organized by the operations used in the puzzle. Clicking on a type of puzzle will lead you to a grid of puzzles organized by size and difficulty. Any link within that grid will take you to a list of puzzles of that size and difficulty level for the operations you selected.
>
> With Nextoy®, **we are pleased to offer a collection of lesson plans around implementing KenKen puzzles in the classroom.** In addition, we are looking forward to developing some PoWs focused on the mathematics and problem-solving behind KenKen, that student KenKen enthusiasts can use to sharpen their skills and practice their mathematical communication. [Bold added for emphasis.]

KenKen puzzles come in a variety of sizes and levels of difficulty. Figure 14.6 gives a 4 x 4 example. The goal is put integers into the small squares (called boxes), subject to a set of rules.

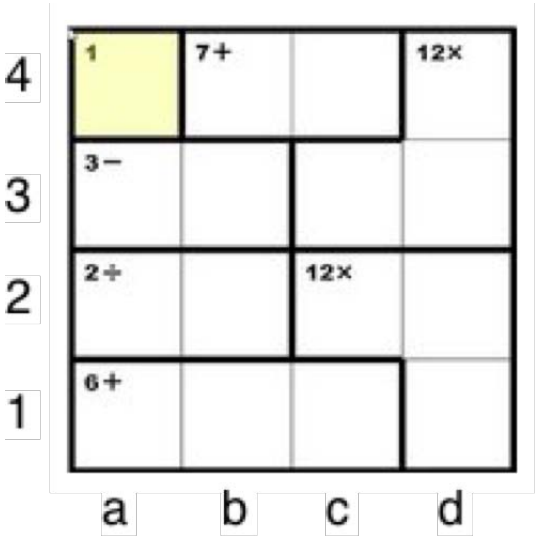


Figure 14.6. A 4 x 4 KenKen puzzle

The rules for playing KenKen are fairly simple:

1. For a 4x4 puzzle, fill in with numbers 1-4.
2. More generally, for an n x n puzzle, fill in with numbers 1, 2, … n.
3. Do not repeat a number in any row. Do not repeat a number in any column.
4. The numbers in each heavily outlined set of squares, called **cages**, must combine to produce the target number in the top left corner of the cage using the mathematical operation indicated.
5. Cages with just one box should be filled in with the target number in the top corner of the box.
6. A number can be repeated within a cage as long as it is not in the same row or column.

In the puzzle in Figure 14.7, the cage in the upper left corner (a4) contains the target number 1 in its upper left corner, and thus can be filled in with the number 1. On its right is a two-box cage. The number in its upper left corner indicates that the sum of the two digits to be placed in this cage is 7. Thus, one of the boxes will contain a 3 and the other will contain a 4. (Can you explain why?) At this stage of solving the puzzle, we don't know which box contains the 3 and which contains the 4.

The two boxes a3 and b3 form a cell, and the number in the upper left cornet of this cell indicates that the difference (larger minus smaller) of the two numbers in this cell is 3. Thus, the two numbers must be 1 and 4. However, the number in box a3 cannot be 1, since there is already a 1 in that column. So, we conclude that a3 contains a 4 and b3 contains a 1. The left part of Figure 14.7 shows the progress we have made so far.

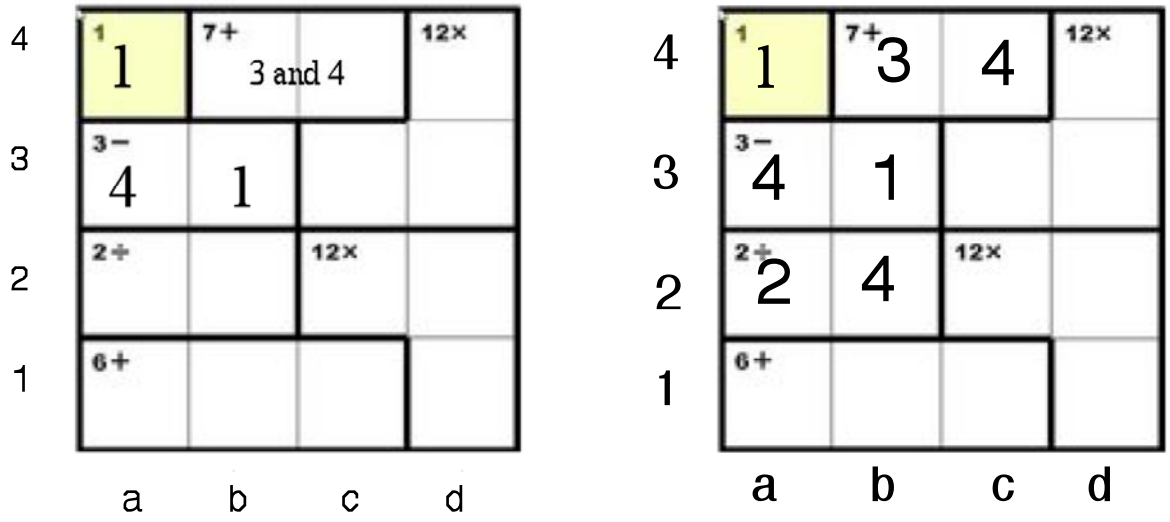


Figure 14.7. Progress in solving puzzle (left) and more progress (right).

We are off to a good start. Suppose that we next take a look at the cell containing the two boxes, a2 and b2. The upper left corner indicates the quotient of the two numbers (larger divided by smaller) is 2. Thus, the numbers must be 4 and 2 or 2 and 1. Based on the boxes we have already filled in, we can eliminate 1 from our possible choices. (Can you explain why?). This leads us to conclude that a2 contains a 2 and b2 contains a 4. That leads us to conclude that b4 contains a 3 and b4 contains a 3. See the right side Figure 14.7. We are now well along toward completing the puzzle.

Sudoku

Sudoku is a very popular one-player puzzle. It can be played using pencil and paper and it can also be played (free) online. See:

> Sudoku (n.d.). Web Sudoku. Retrieved 7/24/2010 from http://www.websudoku.com/. The site contains billions of free puzzles of varying levels of difficulty.
>
> DeSpirt, Debbie (1/1/2007). Solve and create Sudoku puzzles. Retrieved 11/5/2010 from http://www.suite101.com/content/child-sudoku-a11057. Read

more from DeSpirt at Suite101: Child Sudoku: Solve and Create Sudoku Puzzles
http://www.suite101.com/content/child-sudoku-a11057#ixzz14SK3xXYi

Sudoku can be played on various sizes of square playing boards. The game can be played with numbers—the goal being to fill in the missing numbers, subject to a set of rules. The game can also be played with letters—the goal being to fill in the missing letters, subject to a set of rules.

Figure 14.8 shows a two 4 x 4 Sudoku board. The 4 x 4 grid consists of 16 cells and is divided into four 2 x 2 regions. The goal is to fill in all of the empty cells using just digits 1, 2, 3, and 4, and subject to the following rules;

1. Each row is to contain the digits 1–4.
2. Each column is to contain the digits 1–4.
3. Each region is to contain the digits 1–4.

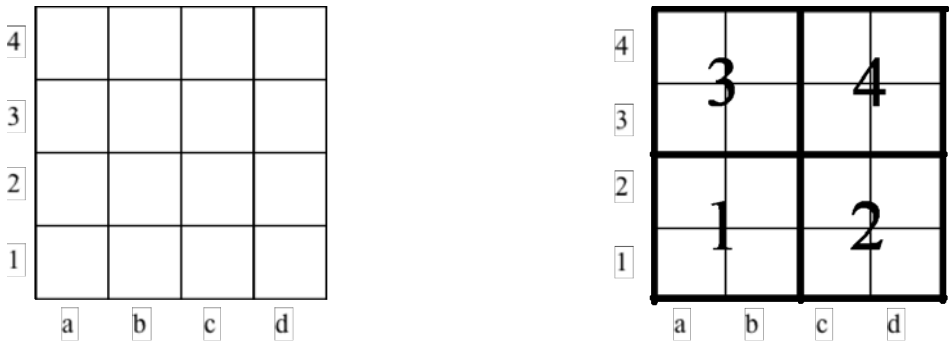


Figure 14.8. 4 x 4 Sudoku board divided into two 2 x 2 regions.

The specific puzzle to be solved begins with some givens—cells that have already been filled in. Figure 14.8 provides an example.

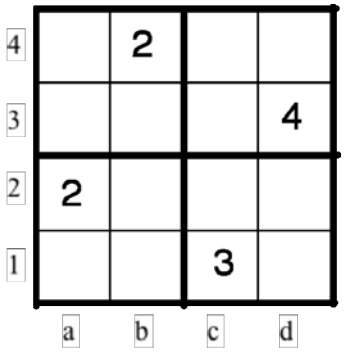


Figure 14.8. A 4 x 4 Sudoku puzzle with four givens.

Study the sample puzzle in Figure 14.8. Notice that the number 2 is a given in cell a2 (in region 1) and in b4 (in region 3) There needs to be a 2 in regions 2 and 4. Careful thought will convince you that the number 2 goes into cells c3 and d1. (Consider, for example, why the cell d4 cannot contain a 2.) By continuing this type of careful thinking, you will be able to complete this puzzle. Figure 14.9 shows part of the progress your authors made along the way.

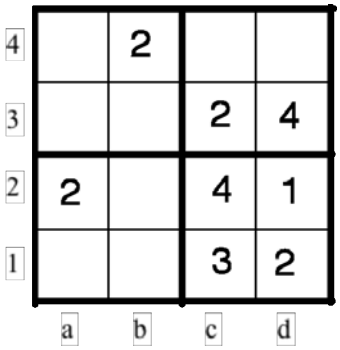


Figure 14.9. Progress toward solving the Sudoku puzzle.

The 4 x 4 Sudoku is quite appropriate for young students. More mathematically mature students will enjoy the challenge of larger board sizes.

KenKen and Sudoku are both quite popular. Both are pure strategy games (pure problem-solving games). They not involve rolling dice or flipping coins to generate moves.

Both have the characteristic of needing to do "look ahead." Of course, our lives are full of the need to do look ahead—to think about the possible consequences of a particular proposed action and to reconcile it with our previous actions. Thus, both of these games—and many other games— provide the opportunity for specific teaching and learning about look ahead.

Remember, when using games in an edutainment manner, that both you and your students should have clear goals in mind both for the educational aspects of the game and for the entertainment aspects of the game. As a teacher, you should make the educational goals explicit and work on transfer of learning from the game to other problem-solving and decision-making situations.

Final Remarks

There are many games that share the dual characteristics of being both fun to play and educationally sound. When working with students, you want to learn what they consider to be fun and you want them to gain personal insights into what they consider to be fun. A particular game may be fun for some students and not fun for other students.

In addition, you want to have quite specific learning goals in mind for your students. These should be made explicit to your students. Your students should be reminded of these learning goals —and, self-assess progress they are making toward these goals—as they play the game.

Each new game should provide students with an opportunity to study their learning processes and to watch themselves gradually gain increased skills in playing the game. It is quite important to help students understand the importance of persistence and that they will get better at games and other learning/problem-solving tasks through their own individual efforts.

Chapter 15: Final Remarks

> "Computers are incredibly fast, accurate, and stupid. Human beings are incredibly slow, inaccurate, and brilliant. Together they are powerful beyond imagination." (Albert Einstein; German and American theoretical physicist; 1879–1955.)

The target audience for this book is educators—teachers, parents, grandparents, and others—who help children learn math. The focus is on the use of math-related games and math-related word problems to engage children in activities that will help to increase their levels of math maturity and their overall levels of math prowess.

The games and the word problems presented in this book can be used over a wide range of grade levels. Some are designed to give students practice in number facts and other routine arithmetical operations. Others are designed to challenge the children's problem-solving skills—to stretch their brains. All are designed to help students increase their levels of math maturity.

Math Maturity

This Math Maturity document contains a number of ideas about what constitutes an increasing level of math maturity, and it presents some ideas about what teachers can be doing to help students increase their levels of math maturity.

Math maturity is a slippery idea. This book discusses the topic in some detail and provides many examples of activities that will help increase math maturity. The focus is on learning that is fun and that is built on understanding, intrinsic motivation, and enthusiastic and self-directed involvement.

However, you will note:

1. There are no references to or good examples of math maturity assessment instruments. The literature in this area is sparse. We need research-based, math-oriented cognitive development instruments that are available in the public domain, easy to use, and designed so the results are useful to both teachers and their students.
2. There are no extensive sets of materials (lesson plans, examples for use at various grade levels, etc.) for use by teachers interested in integrating more math maturity content materials into their math teaching.

Thus, teachers working to increase the math maturity of their students are working at the frontiers of math education—they are advancing the field. As teachers address this task and share their results, they are doing action research. See www.fldoe.org/ese/pdf/**action**-res.pdf.

Math Education Reform

The math reform movement has a strong emphasis on improving math maturity levels of students. It just doesn't make much use of the term *math maturity*.

Here is an example. Earlier in this document there is a list that helps to define math maturity. One of the listed items is "complete the significant shift from learning by memorization to

learning through understanding." Thus, one way to think about the math education wars is that they pit people who support "back to basics, rote memory learning of math" against people who support "learning math in a manner that promotes growth in math maturity." (See http://iae-pedia.org/Math_Education_Wars.) Of course, this is a gross over simplification. However, it suggests that the National Council of Teachers of Mathematics is supportive of teaching for increased math maturity.

For another example, consider the issue of assessment. Math teachers routinely do a "math maturity assessment" of their students. They do this through:

- One-on-one conversation with students. This can be an inherent part of the interaction whenever a student asks for individual help.
- Observing the breadth and depth of answers a student (or, the whole class) gives during class discussions.
- Through analyses of a student's homework and test answers, especially those that require students to show their work and explain their thinking.
- Through reading student journals, if the teacher has students doing math journaling.

Teachers who have a chance to work individually or in very small groups with their students are able to gain deeper insight into their students' levels of math maturity. One of the benefits that a student gains in a one-on-one tutoring situation is that the tutor can identify and then address specific areas of weakness in math maturity.

The Future of Math Maturity Assessment

An individual teacher is concerned about:

1. How can I assess my students' levels of math maturity? I need instruments that I can use and I also need self-assessment instruments that my students can use. As I work to help increase my students' levels of math maturity, I want my students to understand the goals and to be able to measure their own progress toward achieving these goals
2. How can I assess my own level of math content maturing and math pedagogical knowledge maturity? I need an instrument that is relevant to my professional teaching work and that I can use to help me make decisions about my choices of professional development activity.

Each of these is an area needing a substantial amount of research and development. However, a good start can be made without such an investment. Such a start can be made by:

1. Helping teachers to develop a personal professionally useful level of knowledge and understanding about math maturity. This can be occurring in every preservice math course and math methods course designed for preservice and inservice teachers. This book on games in education is an example of materials designed to help teachers.
2. Having teachers explicitly introduce the idea of math maturity to their students and helping their students learn to self-assess. There is quite a lot of research on students learning to do self-assessment and making use of self-assessment. A student can learn to ask and to think about questions such as:

"Do I understand what I am doing and why I am doing it? "Is my understanding just rote memorization, or do I 'really understand' what I am doing?" "What can I do to demonstrate to others that I understand and know what I am doing?"

Math Maturity Ranking Instrument

Math maturity is not a "you have it or you don't have it" type of thing. Rather, think in terms of a number of possible indicators of math maturity, and the progress that a person is making in moving up (increasing math maturity) in these various areas

A teacher evaluating a student at a particular grade level might make use of the simple instrument given in Figure 15.1. As students learn possible meanings of the 11 items, they can learn to self-assess using such an instrument.

Math maturity indicator	**Compared to an average student at this grade level**
1. Math communication	Low (-1) Medium (0) High (+1)
2. Learn to learn math and help others learn math	Low (-1) Medium (0) High (+1)
3. Generalize from specific math example to broad math concept	Low (-1) Medium (0) High (+1)
4. Transfer of math learning	Low (-1) Medium (0) High (+1)
5. Multiple, varied representations of math problems	Low (-1) Medium (0) High (+1)
6. Math problem solving and proofs	Low (-1) Medium (0) High (+1)
7. Math-related word problems in non-math disciplines	Low (-1) Medium (0) High (+1)
8. Math as a human endeavor	Low (-1) Medium (0) High (+1)
9. Math content	Low (-1) Medium (0) High (+1)
10. Mathematical intuition	Low (-1) Medium (0) High (+1)
11. Computers and other math tools	Low (-1) Medium (0) High (+1)

Figure 15.1. A math maturity assessment instrument.

Final Remarks

If a student is not too severely physically or emotionally limited, the student's math maturity steadily increases over time through:

1. General overall increase in cognitive development as her or his brain grows and matures.
2. Learning math in a manner that facilitates higher-order creative and critical thinking, problem solving, theorem proving, communicating in and about math, learning to learn math, and finding fun and joy in using math.
3. Working with math teachers, folk mathematicians (see http://iae-pedia.org/Folk_Math), and others who have a higher level of math maturity

than oneself, and being taught at a level that is a little above one's current level of math maturity.

Math maturity can be increased through good teaching, good instructional materials, and by the active collaboration of learners. Student involvement is an essential component of such math maturity improvement activities. As students gain insights into what constitutes an increasing level of math maturity, they can self assess, reflect on the math they use and do both in school and outside of school, do meta cognition about their math insights, and so on. All of these activities can help a student gain in math maturity

Appendix 1: Make It & Take It, and Blackline Masters

Blackline masters for WordsWorth letter values, Factor Monster tiles, and Pig play sheets. Print on stiff paper and apply scissors or other cutting device. See the next few pages of Appendix 1.

Graph paper. There are many Website that can be used to print graph paper. For example, see http://www.printfreegraphpaper.com/. For printing paper with non-square grids, see http://incompetech.com/graphpaper/.

Handmade Manipulatives Instructions. A recent Internet search on *handmade math manipulatives* produced about 13,000 hits. Here are some examples:

A Math Toolbox in Every Home. http://www.mathcats.com/mathtoolbox/index.html.

Hand Made Manipulative Instructions. http://mason.gmu.edu/~mmankus/Handson/manipulatives.htm.

Large Dice. http://jc-schools.net/tutorials/game/cubelg.pdf.

Small Dice. http://jc-schools.net/tutorials/game/cube.pdf.

Dominoes. Make a rectangular solid with length twice its width, and the third dimension one-third of the width. Use marking pens to color in the dots. Alternatively, just make the thickness of your dominos the thickness of the paper stock you are printing on.

Dominoes Clip Art http://www.misterteacher.com/math/index.html

Dominoes Clip Art http://www.clker.com/search/game+set+tile+bones+domino/1.

Double 9 Dominoes http://www.teachervision.fen.com/tv/printables/scottforesman/Math_1_TTM_3-4.pdf.

WordsWorth Letter Values

a to i letter values

a = 1	b = 2	c = 3	d = 4	e = 5	f = 6	g = 7	h = 8	i = 9

a = 1	b = 2	c = 3	d = 4	e = 5	f = 6	g = 7	h = 8	i = 9

a = 1	b = 2	c = 3	d = 4	e = 5	f = 6	g = 7	h = 8	i = 9

A to I letter values

A = 1	B = 2	C = 3	D = 4	E = 5	F = 6	G = 7	H = 8	I = 9

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A = 1	B = 2	C = 3	D = 4	E = 5	F = 6	G = 7	H = 8	I = 9

a to z letter values

a = 1	b = 2	c = 3	d = 4	e = 5	f = 6	g = 7	h = 8	i = 9
j = 10	k = 11	l = 12	m = 13	n = 14	o = 15	p = 16	q = 17	r = 18
s = 19	t = 20	u = 21	v = 22	w = 23	x = 24	y = 25	z = 26	

A to Z letter values

A = 1	B = 2	C = 3	D = 4	E = 5	F = 6	G = 7	H = 8	I = 9
J = 10	K = 11	L = 12	M = 13	N = 14	O = 15	P = 16	Q = 17	R = 18
S = 19	T = 20	U = 21	V = 22	W = 23	X = 24	Y = 25	Z = 26	

WordsWorth Letter Tiles—Lower Case

a = 1 ▪	b = 2 ▪▪	c = 3 ▪▪▪	d = 4 ▪▪▪▪	e = 5 ▪▪▪▪▪	f = 6 ▪▪▪▪▪▪
g = 7 ▪▪▪▪▪▪▪	h = 8 ▪▪▪▪▪▪▪▪	i = 9 ▪▪▪▪▪▪▪▪▪	j = 10 ▬▬	k = 11 ▬▬ ▪	l = 12 ▬▬ ▪▪
m = 13 ▬▬ ▪▪▪	n = 14 ▬▬ ▪▪▪▪	o = 15 ▬▬ ▪▪▪▪▪	p = 16 ▬▬ ▪▪▪▪▪▪	q = 17 ▬▬ ▪▪▪▪▪▪▪	r = 18 ▬▬ ▪▪▪▪▪▪▪▪
s = 19 ▬▬ ▪▪▪▪▪▪▪▪▪	t = 20 ▬▬ ▬▬	u = 21 ▬▬ ▬▬ ▪	v = 22 ▬▬ ▬▬ ▪▪	w = 23 ▬▬ ▬▬ ▪▪▪	x = 24 ▬▬ ▬▬ ▪▪▪▪
y = 25 ▬▬ ▬▬ ▪▪▪▪▪	z = 26 ▬▬ ▬▬ ▪▪▪▪▪▪				

WordsWorth Letter Tiles—Upper Case

A = 1 ▪	B = 2 ▪▪	C = 3 ▪▪▪	D = 4 ▪▪▪▪	E = 5 ▪▪▪▪▪	F = 6 ▪▪▪▪▪▪
G = 7 ▪▪▪▪▪▪▪	H = 8 ▪▪▪▪▪▪▪▪	I = 9 ▪▪▪▪▪▪▪▪▪	J = 10 ▬▬▬	K = 11 ▬▬▬ ▪	L = 12 ▬▬▬ ▪▪
M = 13 ▬▬▬ ▪▪▪	N = 14 ▬▬▬ ▪▪▪▪	O = 15 ▬▬▬ ▪▪▪▪▪	P = 16 ▬▬▬ ▪▪▪▪▪▪	Q = 17 ▬▬▬ ▪▪▪▪▪▪▪	R = 18 ▬▬▬ ▪▪▪▪▪▪▪▪
S = 19 ▬▬▬ ▪▪▪▪▪▪▪▪▪	T = 20 ▬▬▬ ▬▬▬	U = 21 ▬▬▬ ▬▬▬ ▪	V = 22 ▬▬▬ ▬▬▬ ▪▪	W = 23 ▬▬▬ ▬▬▬ ▪▪▪	X = 24 ▬▬▬ ▬▬▬ ▪▪▪▪
Y = 25 ▬▬▬ ▬▬▬ ▪▪▪▪▪	Z = 26 ▬▬▬ ▬▬▬ ▪▪▪▪▪▪				

Factor Monster Tiles

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20

21	22	23	24	25	26	26	28	29	30
31	32	33	34	35	36	37	38	38	40

41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60

Pig Game Play Sheets

Pig Game play sheet for up to 50 turns or two sets of 25 turns

Name _____ Date _____ Other _____

Turn	Die rolls	Turn	Total	Die rolls	Turn	Total
1						
2						
3						
4						
5						
6						
7						
8						
9						
10						
11						
12						
13						
14						
15						
16						
17						
18						
19						
20						
21						
22						
23						
24						
25						

Pig Game play sheet for up to 25 turns

Name _____ Date _____ Other _____

Turn	Die rolls this turn	Turn score	Total score
			0
1			
2			
3			
4			
5			
6			
7			
8			
9			
10			
11			
12			
13			
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Appendix 2: Some Free Resources

This Appendix contains links to a variety of free resources related to this book.

Math Education Resources

Coin Flipper and Dice Roller Software

Coin Flipper (n.d.). Retrieved 11/6/2010 from http://www.betweenwaters.com/probab/flip/coinmainD.html. Simulates flipping a US quarter. Select **Session** instead of **Historical**, enter the number of flips that you want, and then click on the **Auto Flip** button.

White, Ken (n.d.). Coin Flipper. Retrieved 11/6/2010 from http://shazam.econ.ubc.ca/flip/. Flip one or more Canadian pennies or dimes at Ken White's Coin Flipping Page.

Probability (n.d.). Introduction to probability models. Retrieved 11/6/2010 from http://www.math.csusb.edu/faculty/stanton/m262/intro_prob_models/intro_prob_models.html. Simulates rolling *n*D6 and draws a histogram of the outcome.

The JavaScript Source (n.d.). Games: Dice Roller. Retrieved 11/6/2010 from http://javascript.internet.com/games/dice-roller.html. Choose the number of faces (3, 4, 5, 6, 8, 10, 12, 20, 30, or 100) and the number of dice to roll (1 to 10). The computer produces the sum of the rolls.

Dominoes

Investigating Dominoes. Retrieved 1/1/2011 from http://www.mathwire.com/numbersense/dominoes.html. Many domino activities, including links to double-6 and double-9 domino sets that you can print, domino games, and domino problem-solving.

Dominoes. Retrieved 1/1/2011 from http://lrt.ednet.ns.ca/PD/BLM/pdf_files/number/dominoes.pdf. A pdf file of a complete set of double-9 dominoes. You can print it, laminate it, and cut out the individual dominoes.

Illuminations: Do It with Dominoes. Retrieved 1/1/2011 from http://illuminations.nctm.org/LessonDetail.aspx?ID=U47. In this unit, students explore four models of addition (counting, number line, sets, and balanced equations) using dominoes.

Video

Math Education Free Videos. Retrieved 11/6/2010 from http://iae-pedia.org/Math_Education_Free_Videos. A large collection of math education video materials.

Science Education Free Videos. Retrieved 11/6/2010 from http://iae-pedia.org/Science_Education_Free_Videos. A large collection of science education video materials.

Miscellaneous Other Resources

Brannon Laboratory Duke University. Retrieved 11/6/2010 from http://www.duke.edu/web/mind/level2/faculty/liz/cdlab.htm. Studying the development and evolution of numerical cognition.

Laboratory for Child Development at Johns Hopkins University. Retrieved 11/6/2010 from http://www.psy.jhu.edu/~labforchilddevelopment/.

Math Forum@Drexel. Retrieved 11/6/2010 from http://mathforum.org/. Quoting from the Website:

> The Math Forum is the leading online resource for improving math learning, teaching, and communication since 1992.
>
> - We are teachers, mathematicians, researchers, students, and parents using the power of the Web to learn math and improve math education.
> - We offer a wealth of problems and puzzles; online mentoring; research; team problem solving; collaborations; and professional development. Students have fun and learn a lot. Educators share ideas and acquire new skills.

Mathsisfun. Retrieved 11/6/2010 from http://www.mathsisfun.com/. The site provides assess to a variety of elementary school math games and materials. See, for example, the game Four in a Line at http://www.mathsisfun.com/games/connect4.html. This is a simplified version of Gomoku, traditionally played on a Go board (19 by 19) and requiring 5 in a line.

Science and Numeracy. Retrieved 11/6/2010 from http://literacynet.org/sciencelincs/home.html. Quoting from the site:

> The National Institute for Literacy Science and Numeracy Special Collection provides annotated links to Internet sites that are useful for teaching and learning about science and numeracy. The topics have been arranged according to the national education standards in science and in numeracy.
>
> The collection emphasizes the ways in which science and math skills are important to understanding the world around us.

Units converters. A unit converter provides automatic conversion between a variety of units of measure. Here are two examples from the many different sites available.

http://www.digitaldutch.com/unitconverter/

http://www.unitconversion.org/

Wolfram MathWorld. Retrieved 11/6/2010 from http://mathworld.wolfram.com/. Quoting from the site:

> The Web's most extensive mathematics resource. A free resource from Wolfram Research built with Mathematica technology.

Math-related Games and Puzzles

About.com (n.d.). Math puzzles for kids. Retrieved 4/2/2010 from http://puzzles.about.com/od/familyfun/qt/KidsMath.htm. Quoting from the website:

Improve your math skills with these free online math puzzles and games. Or, print out a customizable math worksheet to test your knowledge of numbers.

Coolmath.com (n.d.). Coolmath-Games. http://www.coolmath-games.com/. A large number of online activities.

Dr. Mike (n.d.). Dr. *Mike's free games.* Retrieved 4/2/2010 from *http://www.dr-mikes-math-games-for-kids.com/index.html*.

Jefferson Lab (n.d.). *Games & puzzles.* Retrieved 4/3/2010 from http://education.jlab.org/indexpages/elementgames.php. Contains a variety of math and science games that can be played online.

KidsKount n.d.). Games from the Netherlands for ages 5–12. Retrieved 5/1/2010 from http://www.fi.uu.nl/rekenweb/en/. A nice collection of online games.

KidZone math. Retrieved 4/2/2010 from http://www.kidzone.ws/math/index.htm. Contains a large number of math education resources organized by grade level and type of resource.

NCTM Illuminations (n.d.). The factor game. Retrieved 7/29/2010 from http://illuminations.nctm.org/LessonDetail.aspx?ID=L253. The game can be played online at the site http://illuminations.nctm.org/ActivityDetail.aspx?ID=12.

NCTM Illuminations (n.d). Fraction Game. Retrieved 9/21/2010 from http://illuminations.nctm.org/ActivityDetail.aspx?ID=18. Play this game online.

Pegg, Ed (n.d.). Ed Pegg Jr.'s *Math Games: A Mathematical Association of America site.* Retrieved 4/2/2010 from http://www.maa.org/news/mathgames.html. These materials are mainly designed for math educators and other adults who have a serious interest in math-related games and some of the history relating to these games.

Word Problems

Background Information

Capital Crest (n.d.). Developing your problem solving skills. Retrieved 1/1/2011 from http://www.crestcapital.com/tax/developing_problem_solving_skills.html.

Internet Classrooms (n.d.). *Taming word problems.* Retrieved 4/2/2010 from http://www.internet4classrooms.com/word_problems_quest.htm. Designed for teachers who want to learn more about helping their students learn about solving word problems. Uses some characters and vocabulary from the Harry Potter book.

Purplemath (n.d.). *Translating word problems into equations.* Retrieved 4/2/2010 from http://www.algebralab.org/lessons/lesson.aspx?file=Algebra_OneVariableWritingEquations.xml. See also http://www.purplemath.com/modules/translat.htm.

Study Guides and Strategies (n.d.). *Solving math word problems.* Retrieved 4/2/2010 from http://www.studygs.net/mathproblems.htm. Quoting from the site:

> Word problems are a series of expressions that fits into an equation. An equation is a combination of math expressions.
>
> There are two steps to solving math word problems:

1. Translate the wording into a numeric equation that combines smaller "expressions."
2. Solve the equation!

Some Sources of Word Problems

PdeagoNet (n.d.). Retrieved 4/2/2010 from http://www.pedagonet.com/brain/brain24.htm.

Math Logic Puzzles and Brain Teasers

Cut The Knot (n.d.). CTK math games for kids. Retrieved 4/16/2010 from http://www.ctkmathgamesforkids.com/. A large collection of games that can be played free online

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> This web page index on Syvum contains FREE online brain teasers and math puzzles at three levels of difficulty—Easy, Medium, and Challenging. All the brain teasers and math puzzles are interactive with immediate scoring to provide continuous learning and entertainment. The brain teasers and math puzzles as well as their explanations use dynamic content.

Appendix 3: Some Not-Free Resources

Math Manipulatives

There are many sources for math manipulatives, and some you can construct for yourself. Browse the Web using search terms such as manipulatives, math manipulatives, math manipulatives list, math manipulatives favorite, math manipulatives kindergarten, virtual manipulatives, and virtual math manipulatives. Here are a few sources that your authors use.

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Math Learning Center. A non-profit company that sells a wide range of math manipulatives and K-8 math curriculum materials. Retrieved 11/6/2010 from http://www.mathlearningcenter.org/.

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My Favorite Math Manipulatives (K–1). Retrieved 4/3/2010 from http://www.kindergarten-lessons.com/math_manipulatives.html.

Nasco. A very wide range of materials such as dice, dominoes, games, base 10 blocks, and so on. Retrieved 4/3/2010 from http://www.enasco.com/math/.

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Index

.....25
ability tests 37
absolute value.....176
abstractness in math 26
action research 34, 188
ADD.....*See* Attention Deficit Disorder
addictive.....93
ADHD.... *See* Attention Deficit Hyperactive Disorder
advance organizers.....71
AI .....*See* Artificial Intelligence
Albrecht, Robert.....2, 8
alert state 21
algorithmic ability.....6
Allison, Dennis 1
alphabetical order.....9
Alzheimer's disease 97
Ansari, Daniel 38
Army Alpha intelligence test 42
Army Beta intelligence test.....42
Artificial Intelligence.....43
Asgari, Mahboubeh.....204
attainment tests.....37
attention.....93
Attention Deficit Disorder 21
Attention Deficit Hyperactive Disorder.....21
Ausubel, David 71
Backgammon 105
base-10 blocks.....5, 24
BASIC.....1
behaviorism.....69
Bentsen, Todd 38, 204
Binet, Alfred 38
board games 5
brain teaser.....47
brain teasers 202
brain theory 20
Brown, John Seely 72, 204
Brown, Stephen I. 31, 204
Bruer, John.....24, 204
Calvin, Duif.....204
casino die 169
Chess.....99
classical conditioning.....73
clearly defined problem 86
cognitive development.....49
cognitive development stage theory.....51
cognitive development test 51
cognitive load.....180
cognitive maturity 22
combinatorics 111
Commons, Michael 51, 204
communication in the language of math 7
competition.....98
components of math maturity 29
Computational Thinking 88
Computer Algebra System 44
computers and math maturity.....33
concrete operations 49
concrete operations stage 27
constructivism 22, 71
constructivist learning 50
convex polyhedron 107
cooperation.....98
counting on.....24
counting words 24
Crace, John.....205
Craps 105
crystallized intelligence.....39, 43
Csikszentmihalyi, Mihaly 94
D10.....5
D6.....5
Danaher-Gilpin, D.....204
De Koven, Bernie.....9
delayed gratification.....7
dementia 97
Denenberg, Larry 29
Descartes, René.....20
developmental psychology.....205
Devlin, Keith.....6, 36, 205
Dewar, Gwen 205
Digit Factory 177
divide and conquer strategy 75
dodecahedron 107
domain.....17
domino tiles.....139
Dominoes 5
Dr. Dobb's Journal.....1
Dr. Mike 201
Dungeons and Dragons 94
Dupuis, Mary 63, 205
dyslexia 44
Einstein, Albert 23
equilateral triangle.....107
Euclid 26
Euclidean Geometry.....26
exhaustive search 16

expected value....127, 167
expertise....97
Factor Monster....151
failure breeds failure....56
fair dice....107
fair die....6
far transfer....73
Fermandez, Alvaro....205
Firedrake, George....1
Flip for Triples....117
fluid intelligence....39, 42
Flynn effect....42, 205
Folk Math....41, 54
folk mathematics....191
formal operations....49
formal operations stage....27
Four 4s Problem....178
Four in a Line....200
Froebel, Friedrich....9
function....16, 17
futurists....125
game
  competition....98
  cooperation....98
  defined....92
  independence....98
Game Theory....126, 129
Games
  Backgammon....105
  Chess....99
  Craps....105
  Digit Factory....177
  Dominos....139
  Dungeons and Dragons....94
  Flip for Triples....117
  Four in a Line....200
  Gift Giver....155
  Go for Low....119, 133
  Hurkle....179
  HurkleQuest on the Number Line....179
  Jeopardy....160
  Klondike....94
  Monopoly....105
  Mother, may I?....34
  Number Race 0 to 12....181
  Pig....160
  Place Value....119
  Poker....92
  Puzzles....97
  Robot....48
  Roll for Twin Doubles....116
  Roll for Two Doubles....116
  Solitaire....92
  Tetris....98
  Try for a Tie....157
  Try for High....119, 133
  Twister....103
  Two-Dice Pig....173
  WordsWorth....9
  WordsWorth in Special Number Land....174
  WordsWorth Plus....12
  WordsWorth Times....176
  Zero Quest....176
Gardner, Howard....40, 43
Gauss, Carl Friedrich....40
Geary, David C....205
general intelligence (g)....39
general math cognitive development scale....52
Geometry cognitive development....52
George Mason University....205
Gift Giver....155
gifted....37
Glbblguy....205
Go for Low....119, 133
Goodheart, E. A.,....204
Graham, Charles....42, 205
guess and check....16
Hagerman, Eric....44, 208
hexahedron....107
Holodeck....95
Huitt, W....50, 205
human endeavor....108
Hummel, J....50, 205
Hurkle....179
HurkleQuest....179
HurkleQuest on the Number Line....179
IAE....*See* Information Age Education
icosahedron....107
independence....98
infinity....25
Information Age Education....1
Information Age Education Newsletter....44
information overload....96
intelligence
  crystallized....39
  fluid....39
Intelligence Quotient....38
International Society for Technology in Education....1
International System of Units....65
intrinsically motivating....93
intuitively obvious....110
IO norming process....39
IQ....*See* Intelligence Quotient
IQ test....51
irrational number....25
ISTE....*See* International Society for Technology in Education
Jefferson, Thomas....137
jeopardy dice games....160, 173
Kaiser Family Foundation....93, 206
Kaufman, David....204
KenKen....90, 182
kinesthetic learning....22
kinesthetic memory pattern....74
Klondike....94

Kronecker, Leopold 19
learning 22
Learning and Leading with Technology 1
learning maturity 21
learning theories 69
learning theory 72
Lehrer, Jonah 206
linguistic ability 6
literacy 10
loaded die 106
Logan, Robert K. 206
look ahead 186
Louisiana Content Standards 206
low-road/high-road theory 74
luck 137
Maier, Eugene 41, 206
math 19
math as a language 7
math cognitive development 36
Math Education Wars 129
Math Forum 22
math manipulatives 5
math maturity 19, 29, 96
components 29
definition 28
Math Maturity Food for Thought 3
math problem solving 7
mathematical expected value 126
mathematically mature adult 3
mathematize 31
mathing 10
math-oriented games 8
math-related help 3
mental model 5
mental modeling 5
mentally challenging 97
Merchant, Linda 63, 205
metacognition 16, 22, 66, 153
Miller, P. M. 204
mind's eye 5, 139, 140
Mitchell, A 206
m-learning. 206
Moniot, Robert K. 151, 206
Monopoly 105
Montana State University 206
Moursund, David 1, 8, 52, 206
Mozart, Wolfgang Amadeus 40
multiple intelligences 40
National Council of Teachers of Mathematics 8
natural language 7
nature 22
nature and nurture 39
NCTM ....*See* National Council of Teachers of Mathematics
near transfer 73
Neisser, Ulric 42
Neo-Piagetians 51
Number Race 0 to 12 181
number sense 6
numeracy 10
numerical ability 6
nurture 22
Official Scrabble® Players Dictionary 14
Ojose, Bobby 49, 207
one-to-one correspondence 10, 24
one-trial learning 72
outcome of the roll 105
Oxford English Dictionary 13
Palov, Ivan 73
Papert, Seymour 92
Pegg, Ed 202
People's Computer Company 1
Perkins, David 36, 43, 74, 207
physical model 109
Piaget, Jean 9, 49
Piaget's 4-stage model 27
Pig 160
Place-value games 119
plasticity 19, 70
Plato 38
Platonic solids 106
play (defined) 92
Plucker, Jonathan 42, 205
poker 92
Polya, George 19, 28, 207
Posit Science 44
Posner, Michael 21, 205
Prensky, Marc 207
Prensky, Mark 93
preoperational 49
preoperational stage 27
problem
defined 81
givens 81
goal 81
ill-defined 82, 86
ownership 81
poorly-defined 86
resources 81
problem posing 146
problem situation 86
problem solving
as a discipline 80
program 48
Project Zero 43, 208
puzzle 97
KenKen 182
Sudoku 184
Puzzles
brainteaser 97
crossword 97
jigsaw 97
KenKen 97

logic 97
Sudoku 97
Pythagoras 25, 107
random integer 5
random numbers 5
range 18
Ratey, John 44, 208
record one's moves strategy 100
reflection 153
reflective intelligence 83
Rice, Grantland 92
Richards, Francis Asbury 204
Robot game 48
Roll for Double Doubles 116
Roll for Two Doubles 116
rote memory 22, 76
Salomon, Gavriel 74, 207
Savill-Smith, C 206
Schoenfeld, Alan H 208
search space 144
Seneca, Lucius 137
sensorimotor 49
sensorimotor stage 27
sensory deprivation tank 20
SI *See* International System of Units
Simon, Theodore 38
situated learning 72
Skinner, Burrhus Frederic 69
solitaire 92
solution space 144
S-R theory ..... *See* Stimulus-Response
stage theory of cognitive development 27
Stanford-Binet test 38
Star Trek 95
STEM 4
Sternberg, Robert 40
Stimulus-Response 73
story problem 78
strategic thinking 126
strategy 16, 119, 120, 153, 164
ask a person 153
break it into smaller pieces 75
build on the previous work of others 75
build on your previous work 75
divide and conquer 16, 75
exhaustive search 16, 145
guess and check 16, 79
look it up 153
record one's moves 100
teaching high-road transfer of learning 75
trial and error 16
subconscious 20
success breeds success 56
Sudoku 90, 184, 185
sustainability 4
table lookup 10, 16
talented 37
Tall, David 26, 208
Tao, Terence 36
target number 35
teaching to the test 96
telling and understanding time 22
Termin, Lewis 41
tests of attainment 37
tetrahedron 107
Tetris 98
theoretical mean score 167
Tiny BASIC 1
tortoise and hare 170
training and education 96
transfer of learning 30, 69, 95
trial and error 16
true coin 109
true die 106
Try for a Tie 157
Try for High 119, 133
Twister 103
Two-Dice Pig 173
van Hiele, Dina and Pierre 52
virtual coin flippers 114
visual thinking 140
Watson, John Broadus 69
Wechsler-Binet 43
Wolpert, Stuart 208
word problem 6, 15, 76, 78
WordsWorth 9
WordsWorth in Special Number Land 174
WordsWorth Plus 12
WordsWorth Times 176
World of Warcraft 103
Yerkes, Robert 41
Yeung, Bernice 208
Zero Quest 176
π 25, 61